Packing Lines, Planes, etc.: Packings in Grassmannian Spaces

J. H. Conway
Department of Mathematics
Princeton University, Princeton, NJ 08544

R. H. Hardin and *N. J. A. Sloane*
Mathematical Sciences Research Center
AT&T Bell Laboratories
Murray Hill, NJ 07974

April 12, 1996

ABSTRACT

This paper addresses the question: how should N n-dimensional subspaces of m-dimensional Euclidean space be arranged so that they are as far apart as possible? The results of extensive computations for modest values of N, n, m are described, as well as a reformulation of the problem that was suggested by these computations. The reformulation gives a way to describe n-dimensional subspaces of m-space as points on a sphere in dimension $(m-1)(m+2)/2$, which provides a (usually) lower-dimensional representation than the Plücker embedding, and leads to a proof that many of the new packings are optimal. The results have applications to the graphical display of multi-dimensional data via Asimov's "Grand Tour" method.

1. Introduction

Although there is a considerable literature dealing with Grassmannian spaces (see for example [6], [29], [45], [28], [20], [18], [46]), the problem of finding the best packings in such spaces seems to have received little attention.

We have made extensive computations on this problem, and have found a number of putatively optimal packings. These computations have led us to conclude that the best definition of distance on Grassmannian space is the "chordal distance" defined in Section 2.

Sections 3, 4, 5, 6 discuss the problems of packing lines in $\mathbb{R}^3$, planes in $\mathbb{R}^4$, n-spaces in $\mathbb{R}^m$, and lines in $\mathbb{R}^m$, respectively.

The results have been placed in the `netlib` archive, and can be accessed there at the address:

`http://netlib.att.com/math/sloane/grass`

Our search has concentrated on packings of $N \leq 55$ subspaces of $G(m,n)$, for $m \leq 16$, $n \leq 3$.

2. Grassmannian space

The *Grassmannian space* $G(m,n)$ is the set of all n-dimensional subspaces of real Euclidean m-dimensional space $\mathbb{R}^m$. This is a homogeneous space isomorphic to $O(m)/(O(n)\times O(m-n))$, and forms a compact Riemannian manifold of dimension $n(m-n)$.

We first discuss how to define the distance between two n-planes P, $Q \in G(m,n)$. The *principal angles* $\theta_1, \ldots, \theta_n \in [0, \pi/2]$ between P and Q are defined by (we follow [19], p. 584)

$$\cos\theta_i = \max_{u \in P} \max_{v \in Q} u \cdot v = u_i \cdot v_i \ ,$$

for $i = 1, \ldots, n$, subject to $u \cdot u = v \cdot v = 1$, $u \cdot u_j = 0$, $v \cdot v_j = 0$ $(1 \leq j \leq i-1)$. The vectors $\{u_i\}$ and $\{v_j\}$ are *principal vectors* corresponding to the pair P and Q.

Wong [45] shows that the *geodesic distance* on $G(m,n)$ between P and Q is

$$d_g(P,Q) = \sqrt{\theta_1^2 + \cdots + \theta_n^2} \ . \tag{2.1}$$

However, this definition has one drawback: it is not everywhere differentiable. Consider the case $n = 1$, for example, and hold one line P fixed while rotating another line Q (both passing through the origin). As the angle ϕ between P and Q increases from 0 to π, the principal angle θ_1 increases from 0 to $\pi/2$ and then falls to 0, and is non-differentiable at $\pi/2$ (see Fig. 1).

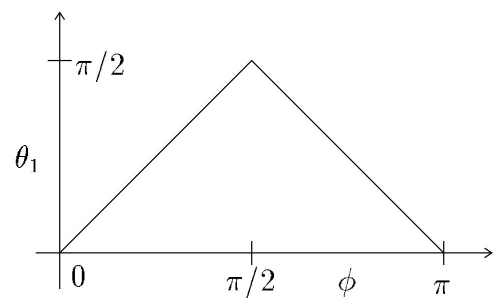

Figure 1: Principal angle θ_1 between two lines as the angle between them increases from 0 to π.

Although one might expect this non-differentiability to be a mere technicality, it does in fact cause considerable difficulties for our optimizer, especially in higher dimensions in cases when many distances fall in the neighborhood of singular points of d_g.

An alternative measure of distance, which we call the *chordal distance*, is given by

$$d_c(P,Q) = \sqrt{\sin^2\theta_1 + \cdots + \sin^2\theta_n} \ . \tag{2.2}$$

The reason for the name will be revealed later. This approximates the geodesic distance when the planes are close, has the property that its square is differentiable everywhere, and, as we shall attempt to demonstrate, has a number of other desirable features.

A third definition has been used by Asimov [3] and Golub and Van Loan [19], p. 584, namely

$$d_m(P,Q) = \theta_n = \max_{i=1,\ldots,n} \theta_i \ .$$

This shares the vices of the geodesic distance.

Of course for $n = 1$ all three definitions are equivalent, in the sense that they lead to the same optimal packings.

We can now state the packing problem: given N, n, m, find a set of n-planes $P_1, \ldots, P_N \in G(m,n)$ so that $\min_{i \neq j} d(P_i, P_j)$ is as large as possible, where d is either geodesic or chordal distance. Since $G(m,n)$ is compact, the problem is well-defined. Because $G(m,n)$ and $G(m,m-n)$ are essentially the same space, we will usually assume that $n \leq m/2$.

We also need some further terminology. A *generator matrix* for an n-plane $P \in G(m,n)$ is an $n \times m$ matrix whose rows span P. The orthogonal group $O(m)$ acts on $G(m,n)$ by right multiplication of generator matrices. The *automorphism group* of a subset $\{P_1, \ldots, P_N\} \subset G(m,n)$ is the subset of $O(m)$ which fixes or permutes these planes.

By applying a suitable element of $O(m)$ and choosing appropriate basis vectors for the planes, any given pair of n-planes P, Q with $n \le m/2$ can be assumed to have generator matrices

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.3}$$

and

$$\begin{bmatrix} \cos\theta_1 & 0 & \cdots & 0 & \sin\theta_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cos\theta_2 & \cdots & 0 & 0 & \sin\theta_2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \cos\theta_n & 0 & 0 & \cdots & \sin\theta_n & 0 & \cdots & 0 \end{bmatrix} \tag{2.4}$$

respectively, where $\theta_1, \ldots, \theta_n$ are the principal angles between them ([45], Theorem 2).

3. Packing lines through the origin in 3-space

Our initial work on this problem was prompted by a question raised in 1992 by Julian Rosenman, an oncologist at the Univ. of North Carolina School of Medicine and Computer Science, in connection with the treatment of tumors using high energy laser beams [39]. He asked for the best way to separate N lines through a given point in $\mathbb{R}^3$, or in other words for the best packings in $G(3,1)$.

Together with W. D. Smith, we had been carrying out an extensive search for the best packings of a given number of points on S^2, i.e. *spherical codes* [24], [25], and we therefore modified our programs to search instead for packings of lines. We omit the details of this search, since we later recalculated these results using the more general methods described in Section 4.

The results are summarized in Table 1, which gives, for each value of N in the range 2 to 55, the minimal angle of the best packing we have found of N lines, and for comparison the minimal angle of the best packing known of $2N$ points on S^2 (taken from [24]). We see that requiring a packing of $2N$ points on S^2 to be antipodal is a definite handicap: only in the cases of 6 and 12 points do the antipodal and unrestricted packings coincide. Decimals in the tables have been rounded to four decimal places.

The last two columns of the table specify the largest automorphism group[1] we have found of any such best antipodal packing of $2N$ points. The fourth column gives the order of the group and the fifth and sixth columns its name in the orbifold notation (cf. [10]) and as the

[1] That is, the subgroup of $O(3)$ that fixes or permutes the $2N$ points.

double cover of a rotation group. The symbol $\pm\mathcal{G}$ indicates that the group consists of the matrices $\pm M$ for $M \in \mathcal{G}$, where $\mathcal{G}$ is a cyclic ($\mathcal{C}$), dihedral ($\mathcal{D}$), tetrahedral ($\mathcal{T}$), octahedral ($\mathcal{O}$) or icosahedral ($\mathcal{I}$) group. In each case the subscript gives the order of the rotation group.

Table 1: Best packings found of N lines through origin in $\mathbb{R}^3$. (The third column gives the best packing known of $2N$ points on a sphere.)

No. of lines N	Min. angle	Min. angle (packing)	Group order	Group name	Group structure	Notes
2	90.0000	109.4712	16	$*224$	$\pm\mathcal{D}_8$	square
3	90.0000	90.0000	48	$*432$	$\pm\mathcal{O}_{24}$	octahedron
4	70.5288	74.8585	48	$*432$	$\pm\mathcal{O}_{24}$	cube
5	63.4349	66.1468	20	$2*5$	$\pm\mathcal{D}_{10}$	pentagonal antiprism (see note)
6	63.4349	63.4349	120	$*532$	$\pm\mathcal{I}_{60}$	icosahedron
7	54.7356	55.6706	48	$*432$	$\pm\mathcal{O}_{24}$	rhombic dodecahedron
8	49.6399	52.2444	2	$\times$	$\pm\mathcal{C}_1$	see note
9	47.9821	49.5567	6	$3\times$	$\pm\mathcal{C}_3$	see note
9	47.9821	49.5567	4	$2*$	$\pm\mathcal{C}_2$	
10	46.6746	47.4310	24	$*226$	$\pm\mathcal{D}_{12}$	hexakis bi-antiprism (see note)
11	44.4031	44.7402	20	$2*5$	$\pm\mathcal{D}_{10}$	
12	41.8820	43.6908	48	$*432$	$\pm\mathcal{O}_{24}$	rhombicuboctahedron
13	39.8131	41.0377	4	$2*$	$\pm\mathcal{C}_2$	
14	38.6824	39.3551	2	$\times$	$\pm\mathcal{C}_1$	
15	38.1349	38.5971	20	$2*5$	$\pm\mathcal{D}_{10}$	see note
16	37.3774	37.4752	120	$*532$	$\pm\mathcal{I}_{60}$	pentakis dodecahedron
17	35.2353	35.8078	2	$\times$	$\pm\mathcal{C}_1$	
18	34.4088	35.1897	6	$3\times$	$\pm\mathcal{C}_3$	
19	33.2115	34.2507	2	$\times$	$\pm\mathcal{C}_1$	
20	32.7071	33.1584	8	$*222$	$\pm\mathcal{D}_4$	
21	32.2161	32.5064	10	$5\times$	$\pm\mathcal{C}_5$	
22	31.8963	31.9834	12	$2*3$	$\pm\mathcal{D}_6$	
23	30.5062	30.9592	2	$\times$	$\pm\mathcal{C}_1$	
24	30.1628	30.7628	24	$3*2$	$\pm\mathcal{T}_{12}$	
25	29.2486	29.7530	6	$3\times$	$\pm\mathcal{C}_3$	
26	28.7126	29.1948	4	$2*$	$\pm\mathcal{C}_2$	
27	28.2495	28.7169	2	$\times$	$\pm\mathcal{C}_1$	

Table 1 (cont.): Best packings found of N lines through origin in $\mathbb{R}^3$.

No. of lines N	Min. angle	Min. angle (packing)	Group order	Group name	Group structure	Notes
28	27.8473	28.1480	2	$\times$	$\pm\mathcal{C}_1$	
29	27.5244	27.5564	28	$2{*}7$	$\pm\mathcal{D}_{14}$	
30	26.9983	27.1928	20	$2{*}5$	$\pm\mathcal{D}_{10}$	
31	26.4987	26.6840	10	$5\times$	$\pm\mathcal{C}_5$	
32	25.9497	26.2350	4	$2*$	$\pm\mathcal{C}_2$	
33	25.5748	25.9474	24	$3{*}2$	$\pm\mathcal{T}_{12}$	
34	25.2567	25.4638	6	$3\times$	$\pm\mathcal{C}_3$	
35	24.8702	25.1709	2	$\times$	$\pm\mathcal{C}_1$	
36	24.5758	24.9265	6	$3\times$	$\pm\mathcal{C}_3$	
37	24.2859	24.4209	2	$\times$	$\pm\mathcal{C}_1$	
38	24.0886	24.1282	2	$\times$	$\pm\mathcal{C}_1$	
39	23.8433	23.9310	6	$3\times$	$\pm\mathcal{C}_3$	
40	23.3293	23.5531	2	$\times$	$\pm\mathcal{C}_1$	
41	22.9915	23.1946	2	$\times$	$\pm\mathcal{C}_1$	
42	22.7075	23.0517	6	$3\times$	$\pm\mathcal{C}_3$	
43	22.5383	22.6744	12	$2{*}3$	$\pm\mathcal{D}_6$	
44	22.2012	22.4679	2	$\times$	$\pm\mathcal{C}_1$	
45	22.0481	22.1540	4	$2*$	$\pm\mathcal{C}_2$	
46	21.8426	22.0276	2	$\times$	$\pm\mathcal{C}_1$	
47	21.6609	21.7221	2	$\times$	$\pm\mathcal{C}_1$	
48	21.4663	21.5206	6	$3\times$	$\pm\mathcal{C}_3$	
49	21.1610	21.3711	2	$\times$	$\pm\mathcal{C}_1$	
50	20.8922	21.0312	2	$\times$	$\pm\mathcal{C}_1$	
51	20.6903	20.8556	2	$\times$	$\pm\mathcal{C}_1$	
52	20.4914	20.6566	2	$\times$	$\pm\mathcal{C}_1$	
53	20.2685	20.4394	2	$\times$	$\pm\mathcal{C}_1$	
54	20.1555	20.3044	6	$3\times$	$\pm\mathcal{C}_3$	
55	20.1034	20.1110	120	$*532$	$\pm\mathcal{I}_{60}$	see note

In some cases the best packings can be obtained by taking the diameters of a known polyhedron, and if so this is indicated in the final column of the table. Other entries in this column refer to the brief descriptions given below.

The entries for $N \leq 6$ were shown to be optimal by Fejes Tóth in 1965 [16] (see also Rosenfeld [37]), and the 7-line arrangement will be proved optimal in Section 5. The solutions for $N \geq 8$ are the best found with over 15000 random starts with our optimizer. There is no guarantee that these are optimal, but experience with similar problems suggests that they will be hard to beat and in any case will be not far from optimal.

For $N = 1, 2, 3, 6$ the solutions are known to be unique, for $N = 4$ there are precisely two solutions ([16], [36], [37]), and for $N = 5, 7, 8$ the solutions appear to be unique. For larger values of N, however, the solutions are often not unique. For $N = 9$ lines there are two different solutions, and in the range $N \leq 30$ the solutions for 10, 22, 25, 27, 29 lines (and possibly others) contain lines that "rattle", that is, lines which can be moved freely over a small range of angles without affecting the minimal angle.

Notes on Table 1

$N = 5$. Five of the six diameters of a regular icosahedron.

$N = 8$. The putatively optimal arrangement forms an unpleasant-looking configuration of 16 antipodal points on S^2, with no further symmetry. The convex hull is shown in Fig. 2. In contrast, the putatively best packing of 16 points (Fig. 3) has a group of order 16.

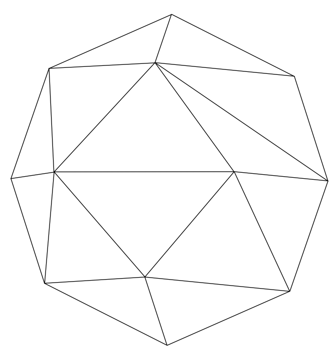

Figure 2: Best antipodal packing found of 16 points.

$N = 9$. There appear to be two inequivalent solutions. The nicest has symmetry group $3\times$, of order 12 (or $[2^+, 6^+]$ in Coxeter's notation [13]) — see Fig. 4. The points lie in six horizontal layers of equilateral triangles. The nine points in the Northern hemisphere can be located by drawing seven equilateral spherical triangles of edge length 47.98213264°, arranged as in Fig. 5. The North pole is located at the midpoint of the central triangle. The second solution, shown in Fig. 6, has group $2*$, of order 4.

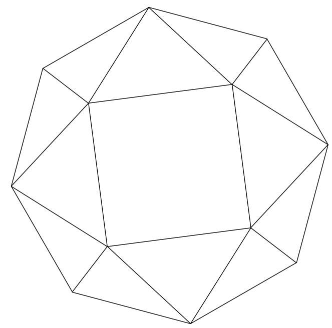

Figure 3: Best (unrestricted) packing of 16 points known.

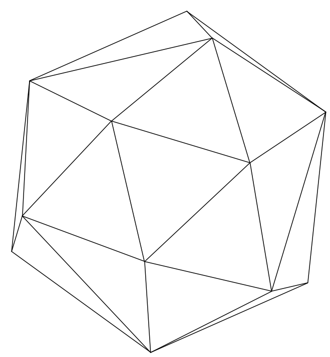

Figure 4: Antipodal packing of 18 points with group of order 6.

$N = 10$. This is a "hexakis bi-antiprism", since it consists of a bi-(hexagonal antiprism) together with an axial line that can "rattle", giving infinitely many solutions. The axis appears horizontally in Fig. 7.

$N = 15$. Combinatorially, this is an "axially depleted pentakis dodecahedron", obtained from a pentakis dodecahedron (the solution for $N = 16$) by omitting two opposite vertices (see Fig. 8). However, the angular separation is slightly greater than for $N = 16$.

$N = 55$. The 110 antipodal points can be taken to be the union of the vertex sets of a dodecahedron (20), an icosidodecahedron (30) and a truncated icosahedron (60) — see Fig. 9. The 15 lines through the icosidodecahedral points "rattle". This arrangement of points can be obtained from a "geodesic dome" or 12-fold reticulated icosahedron [12] by omitting its twelve pentagonal vertices, and so may be described as consisting of the vertices of a "depleted 12-fold reticulated icosahedron". Incidentally, our best solution for 61 lines has a symmetry group of

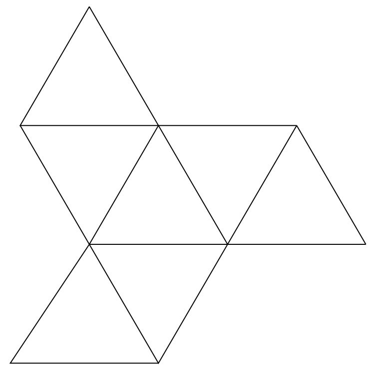

Figure 5:

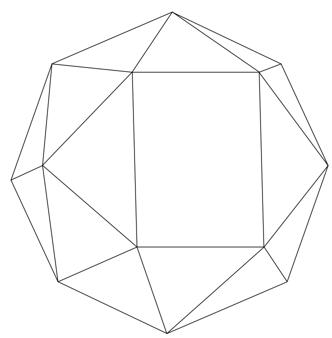

Figure 6: Antipodal packing of 18 points with group of order 4.

order 6, and is not found by placing points at the centers of the pentagonal faces of Fig. 9.

Packings of lines in higher dimensional spaces will be discussed in Section 6.

4. Packing two-dimensional planes in $\mathbb{R}^4$

In [3] Asimov proposed a technique called the "Grand Tour" for displaying multi-dimensional data on a two-dimensional computer screen. His idea was to choose a finite sequence or "tour" of two-dimensional planes that are in some sense "representative" of all planes, and to project the data onto each plane in turn, in the hope that the viewer will be able to notice any pattern or structure that is present. This technique has been implemented in the XGobi program [5], [43].

In 1993 Dianne Cook asked us if we could modify our algorithm for finding spherical codes, in order to search for packings in $G(m, 2)$ for $m \geq 4$. Furthermore, for the Grand Tour application, there should be a Hamiltonian circuit through the planes, in which edges indicate neighboring planes. We accomplished this in the following way.

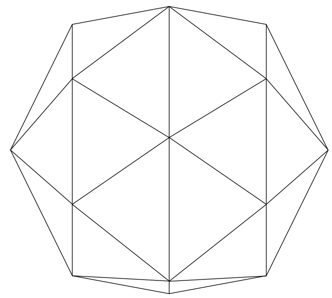

Figure 7: Best antipodal packing found of 20 points.

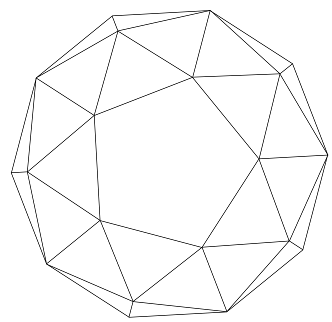

Figure 8: Best antipodal packing found of 30 points.

Following the methods we had used to find packings in S^d and experimental designs in various spaces (see [22], Sec. 4.11; [23], [25]), corresponding to a set of planes $S = \{P_1, \ldots, P_N\} \subseteq G(m,n)$ we define a potential

$$\Phi_c(S) = \sum_{i<j} \frac{1}{d_c(P_i, P_j) - A} ,$$

where A is a suitably chosen constant. There is a similar definition for Φ_g involving d_g. Initially A is set to 0 and S is a randomly chosen set of planes. We invoke the Hooke-Jeeves "pattern search" optimization algorithm to modify S, attempting to minimize $\Phi_c(S)$. After a fixed time (we used 100 steps of the optimizer), A is advanced halfway to the current minimal distance between the P_i, and the process is repeated, terminating when no further improvement is obtained to the accuracy of the machine. The whole process is repeated with several thousand random starts — and also with initial configurations taken from other sources, such as subsets or supersets of other arrangements — and the best final configuration is recorded.

The partial derivatives of $\Phi_c(S)$ were found analytically, but for $\Phi_g(S)$ we calculated them by numerical differentiation.

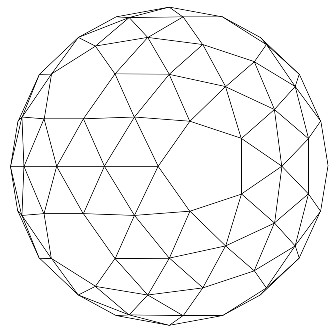

Figure 9: Best antipodal packing found of 110 points.

We began by computing a table of packings of N planes in $G(4,2)$, for $N \leq 50$, with respect to both d_c and d_g. The results for packings of $N \leq 24$ planes are summarized in Table 2. Columns 2 and 3 refer to the best packings found for the chordal distance, and give both

$$d_c^2 = \min_{i \neq j} d_c^2(P_i, P_j) \quad \text{and} \quad d_g^2 = \min_{i \neq j} d_g^2(P_i, P_j)$$

for such packings. Columns 4 and 5 refer similarly to the best packings found with respect to geodesic distance.

Understanding these results was hindered by the fact that planes are, as the name suggests, *plain*, with no distinguishing features, and when produced by the computer appear as random generator matrices referred to a random coordinate frame. However, we found that the set of principal vectors in the planes could often be used to find a coordinate system that would reveal some of the structure of the planes.

For two planes the best packing is the same for both metrics, and consists of two mutually orthogonal planes with principal angles $\pi/2$, $\pi/2$, so $d_c^2 = 2$, $d_g^2 = \pi^2/2$.

For three planes the two answers are different. The best packing for chordal distance (Fig. 10a) has principal angles $\pi/3$, $\pi/3$ between each pair of planes, so $d_c^2 = 3/2$, $d_g^2 = 2\pi^2/9$, whereas the best packing for geodesic distance (Fig. 10b) has angles $\pi/6$, $\pi/2$, so $d_c^2 = 5/4$, $d_g^2 = 5\pi^2/18$.

Postponing discussion of $N = 4$ and 5 for the moment, let us consider the case of six planes, where we discovered that the answer for both metrics formed a regular simplex[2]: the principal angles between every pair of planes were $\arcsin 1/\sqrt{5}$ and $\pi/2$. This set of planes is

[2]Although not an isoclinic configuration, cf. Wong [44]. There seems to be only a slight overlap connection between that problem and ours.

Table 2: Comparison of putatively best packings of N planes in $G(4,2)$ with respect to d_c and d_g.

	Best w.r.t. d_c		Best w.r.t. d_g	
N	d_c^2	d_g^2	d_c^2	d_g^2
2	2.0000	4.9348	2.0000	4.9348
3	1.5000	2.1932	1.2500	2.7416
4	1.3333	1.8253	1.2000	2.6824
5	1.2500	1.9739	1.2000	2.6824
6	1.2000	2.6824	1.2000	2.6824
7	1.1667	1.6440	0.9875	2.1281
8	1.1429	1.5818	0.9700	1.9235
9	1.1231	1.5175	0.9501	1.8087
10	1.1111	1.5725	0.9764	1.7886
11	1.0000	1.2715	0.9247	1.6711
12	1.0000	1.3413	0.9204	1.6416
13	1.0000	1.2348	0.9133	1.5600
14	1.0000	1.2337	0.8933	1.5327
15	1.0000	1.2337	0.8923	1.5284
16	1.0000	1.2337	0.8904	1.5210
17	1.0000	1.2337	0.8549	1.3925
18	1.0000	1.2337	0.8504	1.3768
19	0.9091	1.1666	0.8412	1.3477
20	0.9091	1.1666	0.8351	1.3284
21	0.8684	1.0352	0.8225	1.2834
22	0.8629	1.0592	0.8046	1.2385
23	0.8451	1.0081	0.7910	1.2012
24	0.8372	0.9901	0.7812	1.1707

conveniently described using simplicial coordinates. Let A, B, C, D, E be the vectors from the center of a regular simplex in $\mathbb{R}^4$ to its vertices, with $A+B+C+D+E=0$, and write $[a,b,c,d,e]$ for $aA+bB+cC+dD+eE$. Then one of the six planes is spanned by $[1,\tau,1,0,0]$ and its cyclic shifts, where $\tau=(1+\sqrt{5})/2$, and the other five planes are obtained from it by taking even permutations of these coordinates. (We later found an equally good packing with respect to chordal distance that was not a simplex with respect to geodesic distance. This is described below.)

The six planes intersect the surface of the unit ball of $\mathbb{R}^4$ in a remarkable link, the "Hexalink". It can be shown that apart from the n-component unlinks and Hopf links, which exist for all n, the Hexalink is the only link of circular rings in which there are orientation-preserving symmetries taking any two links to any other two. (Details will be given elsewhere.)

1	0	1	0
0	1	0	1
1	0	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
0	1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
1	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
0	1	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

(a)

1	0	0	0
0	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$
0	1	0	0
0	0	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
0	0	1	0
$-\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{2}}$

(b)

Figure 10: Generator matrices for best packings of three planes in $G(4,2)$ with respect to (a) chordal and (b) geodesic distance.

The discovery of this 6-vertex simplex was initially somewhat of a surprise, since $G(4,2)$ is only a 4-dimensional manifold. It did suggest that the answer should somehow be related to the six equi-angular diameters of the icosahedron, and led to the following reformulation of the problem.

We remind the reader that any element α of $SO(4)$ may be represented as

$$\alpha : x \mapsto \bar{\ell} x r \ ,$$

where $x = x_0 + x_1 i + x_2 j + x_3 k$ represents a point on S^3 and ℓ, r are unit quaternions [15]. The pair $-\ell$, $-r$ represent the same α. The correspondence between α and $\pm(\ell, r)$ is one-to-one.

Given a plane $P \in G(4,2)$, let α be the element of $SO(4)$ that fixes P and negates the points of the orthogonal plane $P^{\perp}$. Then $\alpha^2 = 1$, and for this α, it is easy to see that $\ell = \ell_1 i + \ell_2 j + \ell_3 k$ and $r = r_1 i + r_2 j + r_3 k$ are purely imaginary unit quaternions. This establishes the following result (which can be found for example in Leichtweiss [29]).

Theorem 1. *A plane $P \in G(4,2)$ is represented by a pair $(\ell, r) \in S^2 \times S^2$, with $(-\ell, -r)$ representing the same P.*

There are simple formulae relating P and (ℓ, r), pointed out to us by Simon Kochen. Given (ℓ, r), if $\ell \neq -r$ then P is spanned by the vectors corresponding to the quaternions $u = 1 - \ell r$ and $v = \ell + r$. The special case when $\ell = r$ is even simpler: take $u = 1$, $v = \ell$. If $\ell = -r$, take u and v to be purely imaginary unit quaternions such that ℓ, u, v correspond to a coordinate frame. Conversely, if P is spanned by two orthogonal unit vectors represented by quaternions u, v, then $\pm(\ell, r) = \pm(u\bar{v} - v\bar{u}, \bar{v}u - \bar{u}v)$.

Given two planes $P, Q \in G(4,2)$, represented by $\pm(\ell, r)$, $\pm(\ell', r')$, respectively, the principal angles θ_1, θ_2 between them may be found as follows. Let ϕ (resp. ψ) be the angle between ℓ

and ℓ' (resp. r and r'), with $0 \le \phi, \psi \le \pi$. If $\phi + \psi > \pi$, replace ϕ by $\pi - \phi$ and ψ by $\pi - \psi$, so that $0 \le \phi + \psi \le \pi$, with $\phi \le \psi$ (say). Then

$$\theta_1, \theta_2 = \frac{\psi \pm \phi}{2} \ , \tag{4.1}$$

$$d_g^2(P,Q) = \frac{\psi^2 + \phi^2}{2} \ , \tag{4.2}$$

$$d_c^2(P,Q) = 1 - \cos\psi \ \cos\phi \ . \tag{4.3}$$

We omit the elementary proof.

A set $S = \{P_1, \ldots, P_N\} \subseteq G(4,2)$ is thus represented by a "binocular code" consisting of a set of pairs $\pm(\ell_i, r_i) \in S^2 \times S^2$. We call the list of $2N$ points $\pm\ell_i$ (they need not be distinct) the "left code" corresponding to S, and the points $\pm r_i$ the "right code". Conversely, given two multisets $L \subseteq S^2$, $R \subseteq S^2$, each of size $2N$ and closed under negation, and a bijection or "matching" f between them that satisfies $f(-\ell) = -f(\ell)$, $\ell \in L$, we obtain a set of N planes in $G(4,2)$.

The binocular codes for the d_c-optimal packings of $N = 2, \ldots, 6$ planes are shown in Figs. 11, 12. Except for $N = 3$, the left and right codes are identical. Matching points from the left and right codes are labeled with the same symbol. For $N \le 5$ the points lie in the equatorial plane, and for $N \le 4$ there are repeated points. The points lie on regular figures, except for $N = 4$ where the points are $\pm(1,0,0)$, $\pm\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, 0\right)$.

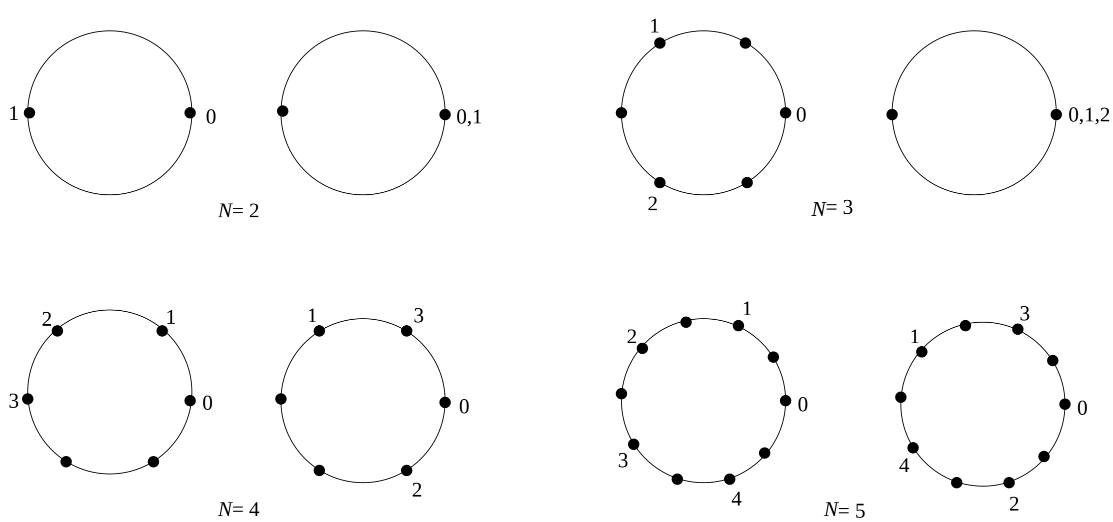

Figure 11: Binocular codes describing best packings of $N = 2, \ldots, 5$ planes in $G(4,2)$ for chordal distance.

For $N = 6$ the left and right codes consist of the 12 vertices of an icosahedron (Fig. 12). Let these be the points

$$\lambda(0, \pm 1, \pm\tau),\ \lambda(\pm\tau, 0, \pm 1),\ \lambda(\pm 1, \pm\tau, 0)\ , \tag{4.4}$$

where $\lambda = 1/\sqrt{\tau + 2}$. The matching is obtained by mapping each point to its algebraic conjugate (i.e. replacing $\sqrt{5}$ by $-\sqrt{5}$), and rescaling so the points again lie on a unit sphere. As already mentioned, the principal angles between each pair of these planes are arcsin $1/\sqrt{5}$ and $\pi/2$, so $d_c^2 = 6/5$, $d_g^2 = 2.6824$.

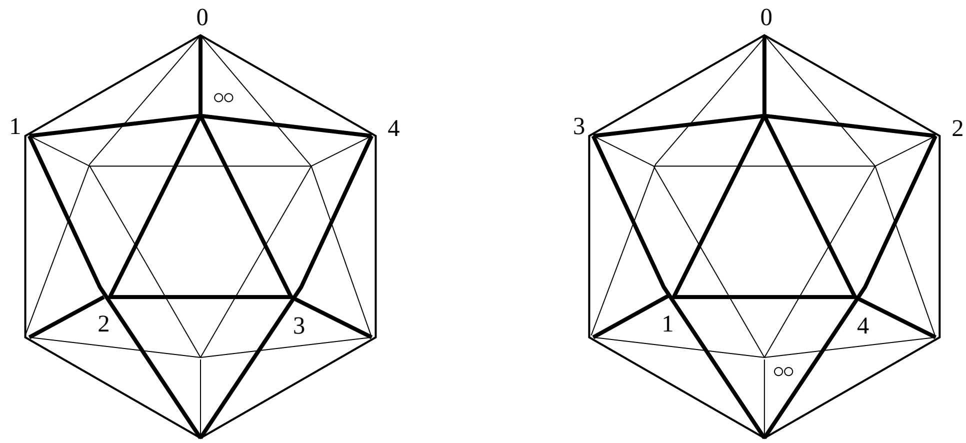

Figure 12: Best packing of 6 planes in $G(4, 2)$ with respect to both metrics. Left and right codes comprise vertices of icosahedron. Adjacent vertices in one code are matched with non-adjacent vertices in the other code.

There is a second set of six planes with $d_c^2 = 6/5$, but with d_g^2 only equal to 2.0030. Here the left and right codes form the vertices of what we shall call the "anti-icosahedron", consisting of the points $\lambda(0, \pm 1, \pm\tau)$, $\lambda(\pm\tau, 0, \pm 1)$, $\lambda(\pm\tau, \pm 1, 0)$ (compare (4.4)), and shown in Fig. 13. Topologically this is a "parallel bi-slit cuboctahedron", obtained by dividing two opposite square faces of a cuboctahedron into two triangles by parallel lines. The matching sends $\lambda(0, \pm 1, \tau)$ to $\lambda(0, \pm\tau, 1)$, $\lambda(\tau, 0, \pm 1)$ to $\lambda(1, 0, \pm\tau)$ and $\lambda(\pm\tau, 1, 0)$ to $\lambda(\pm 1, \tau, 0)$.

Four larger putatively d_c-optimal packings that the computer found are also worth mentioning, those of 10, 18, 48 and 50 planes.

For the 10-plane packing the left and right codes consist of the vertices of a decagonal prism with coordinates

$$\pm\left(\sqrt{\frac{2}{3}}\cos\ r\theta,\ \sqrt{\frac{2}{3}}\sin\ r\theta,\ \sqrt{\frac{1}{3}}\right),\quad r = 0, \ldots, 9\ ,$$

where $\theta = \pi/5$. A typical point $(\sqrt{2/3}\cos\ r\theta,\ \sqrt{2/3}\sin\ r\theta,\ \sqrt{1/3})$ in the left code is matched with $(-1)^r\ (\sqrt{2/3}\cos 3r\theta,\ \sqrt{2/3}\sin 3r\theta,\ \sqrt{1/3})$ in the right code. Then $d_c^2 = 10/9$, $d_g^2 =$

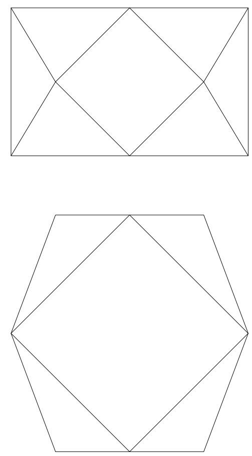

Figure 13: Top and side views of anti-icosahedron.

1.5725. The most interesting property of this configuration is that, although three different sets of canonical angles occur, the chordal distance between every pair of planes is the same — this is a regular simplex!

We find it surprising that this is superior to the packing of ten planes obtained by matching the vertices of a dodecahedron to their algebraic conjugates, as shown in Figure 14.

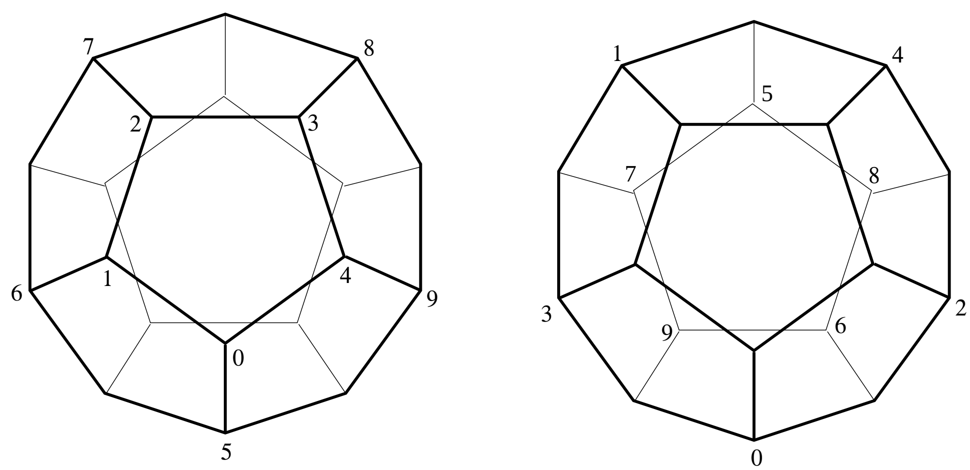

Figure 14: Matching of vertices of dodecahedron.

18 planes. Let $\mathcal{O}$ denote the set of vertices of a regular octahedron. The binocular code for the 18-plane packing is $\mathcal{O} \times \mathcal{O} := \{(\ell, r) : \ell, r \in \mathcal{O}\}$. Projectively, the 18 two-spaces are the edges of a desmic triple[3] of tetrahedra [42], [27], [11]. We were pleased to see this configuration

[3] Two tetrahedra are called *desmic* if they are in perspective from four distinct points — these form a third tetrahedron desmic with either. The 18 edges of such a desmic triple are also the edges of a unique other desmic triple.

appear, since it was already familiar in the context of quantum logic [7]. This set of planes can be constructed directly by taking the planes spanned by all pairs of mutually perpendicular minimal vectors of the D_4 lattice (cf. [9]). In this form the planes have generator matrices such as

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & - \end{bmatrix} \text{ (12 planes)}, \quad \begin{bmatrix} + & + & 0 & 0 \\ + & - & 0 & 0 \end{bmatrix} \text{ (6 planes)} .$$

Three different sets of principal angles occur, 0, $\pi/2$; $\pi/4$, $\pi/4$; and $\pi/2$, $\pi/2$; so that $d_c^2 = 1$, $d_g^2 = 1.2337$. The automorphism group of this packing has structure $[3,4,3].2$ and order 2304. These 18 planes can also be described as the set of planes that meet the 24-cell in one of its equatorial squares.

48 and 50 planes. Let $\mathcal{C}$ denote the set of vertices of a cube. Then the binocular codes for the 48- and 50-plane packings are respectively $\mathcal{O} \times \mathcal{C} \cup \mathcal{C} \times \mathcal{O}$ and $\mathcal{O} \times \mathcal{O} \cup \mathcal{C} \times \mathcal{C}$. For both arrangements we have $d_c^2 = 2/3$, $d_g^2 = 0.7576$. Both are believed to be optimal. (We mention the 48-plane packing because of its greater symmetry.)

The matching problem. In order to maximize the minimal chordal distance between the planes, the matching between the left and right codes should (from (4.3)) be chosen so as to minimize the maximal value of $\cos\psi \cos\phi$. Stated informally, the matching should be such that if two points are close together on one of the spheres then the points to which they are matched should be far apart on the other sphere. In the case of the icosahedron, for example (see Fig. 12), the matching sends adjacent vertices to non-adjacent vertices. There is a unique way to do this.

At this point we were tempted to see if any new record packings in $G(4,2)$ could be obtained by taking the $2N$ antipodal points corresponding to a good N-line packing in $G(3,1)$, and matching them with themselves in an optimal way. However, it was not easy to see how to solve the matching problem. Fortunately David Applegate (personal communication) found that it could be reformulated as an integer programming problem, as follows.

Given an antipodal set $S = \{P_1, \ldots, P_{2N}\} \subseteq S^2$, we wish to find a permutation f of S with the property that $f(-P_i) = f(P_i)$, for all i, and such that the minimal value of $1 - P_i \cdot P_j f(P_i) \cdot f(P_j)$ $(i \neq j)$ is maximized. If we do the maximization by binary search, we may define $\pi(P_i, P_j) = 1$ if $f(P_i) = P_j$, or 0 otherwise, with the constraints

$$\pi(P_i, P_j) \quad = \quad \pi(-P_i, -P_j) \ ,$$

$$\sum_{j=1}^{2N} \pi(P_i, P_j) = 1, \quad 1 \le i \le 2N ,$$
$$\sum_{i=1}^{2N} \pi(P_i, P_j) = 1, \quad 1 \le j \le 2N ,$$

and

$$\pi(P_i, P_j) + \pi(P_k, P_\ell) < 1 \quad \text{for all} \quad 1 \le i, j, k, \ell \le 2N \quad \text{such that} \quad 1 - P_i \cdot P_k \; P_j \cdot P_\ell < M .$$

There is a feasible solution if and only if $d_c^2 \ge M$.

Applegate kindly implemented this procedure, and used it to solve the matching problem for our best packings in $G(3,1)$ and for various polyhedra. Unfortunately no new records have yet been obtained by this method.

Best packings for the geodesic metric. So far we have mostly discussed packings that attempt to maximize the minimal chordal distance between planes. We also computed packings for the geodesic distance, and we shall now describe some of them. In general, however, these are much less symmetric than the chordal-distance packings, especially for more than 16 planes.

For $N = 4, 5$ the solutions are subsets of the 6-plane packing. For $N = 7$ the left code is a heptagonal anti-prism and the right code is an equatorial 14-gon.

For $N = 12$ the binocular code is the set $\{\pm(\ell, r)\}$, where ℓ (resp. r) runs through the vertices of a regular tetrahedron (resp. equatorial equilateral triangle).

For $N = 16$ the left and right codes are the union of two similarly oriented square prisms. For larger N the geodesic packings do not seem so interesting. For $N = 18$, for example, the best packing has no non-trivial symmetries.

Hamiltonian paths. One further question remains to be discussed, that of arranging the planes in a circuit in such a way that adjacent planes are close together, for the "Grand Tour" application. This turned out to be a much easier problem than finding the packings. We handled it in two different ways. For configurations such as the 48-plane arrangement, where there was an obvious notion of adjacency — in this case, defining two planes to be adjacent if the principal angles are $\pi/4$, $\pi/4$ — we represent the packing by a graph with nodes representing the planes, and look for a Hamiltonian cycle. In less regular cases, we convert the packing into a traveling salesman problem using chordal distance to define the distance between nodes, and look for a minimal length circuit. In both cases we were able to make use of the travelling salesman programs of Applegate et al. [2], which can handle 100-node graphs without difficulty. Some of the packings in the `netlib` archive mentioned in Sect. 1 (those with suffix `.ham`) have been arranged in cycles in this way.

The above reformulation in terms of binocular codes applies only to $G(4,2)$ (we realized from the beginning that this case would be special, since $G(4,2)$ is the only Grassmannian space where the Riemannian metric is not unique [29]). In the next section we describe a second reformulation that applies to the general case.

5. Packing n-planes in $\mathbb{R}^m$

Three observations contributed to the second reformulation.

(i) We noticed (see Table 2) that for several examples of N-plane packings in $G(4,2)$ the largest value of d_c^2 that we could attain was $N/(N-1)$, and that in every case this was an upper bound. Further experimentation with other packings in $G(m,n)$ for $m \leq 8$ led us to guess an upper bound of

$$d_c^2 \leq \frac{n(m-n)}{m} \cdot \frac{N}{N-1} , \tag{5.1}$$

which again we could achieve for some small values of N. The form of (5.1) was suggestive of the Rankin bound for spherical codes [9] or the Plotkin bound for binary codes [33].

(ii) Investigation of the 18-plane chordal-distance packing revealed that, with respect to chordal distance, this has the structure of a regular orthoplex (or generalized octahedron) with 18 vertices. Combining this with the fact that the 10-plane packing formed a regular simplex, we had strong evidence that $G(4,2)$ should have an isometric embedding (with respect to d_c) into $\mathbb{R}^9$.

(iii) A computer program was therefore written to determine the lowest dimensions into which our library of packings in $G(m,n)$ could be isometrically embedded. More precisely, for a given set of N planes in $G(m,n)$, we searched for the smallest dimension D such that there are N points in $\mathbb{R}^D$ whose Euclidean distances coincide with the chordal distances between our planes.

The results were a surprise: it appeared that $G(m,n)$ with chordal distance could be isometrically embedded into $\mathbb{R}^D$, for $D = \binom{m+1}{2} - 1$, independent of n. Furthermore the points representing elements of $G(m,n)$ were observed to lie on a sphere of radius $\sqrt{n(m-n)/2m}$ in $\mathbb{R}^D$.

Aided by discussions with Colin Mallows, we soon found an explanation: just associate to each $P \in G(m,n)$ the orthogonal projection map from $\mathbb{R}^m$ to P. If A is a generator matrix for P whose rows are orthogonal unit vectors, then the projection is represented by the matrix

$$\mathcal{P} = A^{tr} A \ . \tag{5.2}$$

$\mathcal{P}$ is an $m \times m$ symmetric idempotent matrix, which is independent of the particular orthonormal generator matrix used to define it. Changing to a different coordinate frame in $\mathbb{R}^m$ has the effect of conjugating $\mathcal{P}$ by an element of $O(m)$. With the help of (2.3), we see that

$$\text{trace } \mathcal{P} = n \ . \tag{5.3}$$

Thus $\mathcal{P}$ lies in a space of dimension $\binom{m+1}{2} - 1$.

Let $\|\ \|$ denote the L_2-norm of a matrix: if $M = (M_{ij})$, $1 \le i,j \le m$,

$$\|M\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} M_{ij}^2} = \sqrt{\text{trace } M^{tr} M} \ .$$

For $P, Q \in G(m,n)$, with orthonormal generator matrices A, B, and principal angles $\theta_1, \ldots, \theta_n$, an elementary calculation using (2.3), (2.4) shows that

$$\begin{aligned} d_c^2(P,Q) &= n - (\cos^2\theta_1 + \cdots + \cos^2\theta_n) \\ &= n - \text{trace } A^{tr} A B^{tr} B \\ &= \frac{1}{2}\|\mathcal{P} - \mathcal{Q}\|^2 \ , \end{aligned} \tag{5.4}$$

where $\mathcal{P}$, $\mathcal{Q}$ are the corresponding projection matrices.

Note that if we define the "de-traced" matrix $\bar{\mathcal{P}} = \mathcal{P} - \frac{n}{m} I_m$, then trace $\bar{\mathcal{P}} = 0$, and $\|\bar{\mathcal{P}}\|^2 = \frac{n(m-n)}{n}$. We have thus established the following theorem.

Theorem 2. *The representation of n-planes $P \in G(m,n)$ by their projection matrices $\bar{\mathcal{P}}$ gives an isometric embedding of $G(m,n)$ into a sphere of radius $\sqrt{n(m-n)/n}$ in $\mathbb{R}^D$, $D = \binom{m+1}{2} - 1$, with $d_c(P,Q) = \frac{1}{\sqrt{2}}\|\mathcal{P} - \mathcal{Q}\|$.*

Thus chordal distance between planes is $1/\sqrt{2}$ times the straight-line distance between the projection matrices (which explains our name for this metric). The geodesic distance between the planes is $1/\sqrt{2}$ times the geodesic distance between the projection matrices, measured along the sphere in $\mathbb{R}^D$.

Figure 5 attempts to display the embeddings of $G(m,0)$, $G(m,1),\ldots,G(m,m)$ in $\mathbb{R}^{D+1}$. Since $\|\mathcal{P}-\frac{1}{2}I_m\|^2 = m/4$, all the planes lie on the large sphere, centered at $\frac{1}{2}I_m$, of radius $\sqrt{m}/2$. The members of $G(m,n)$ lie on the intersection of the large sphere with the plane $\text{trace}(\mathcal{P}) = n$, which intersection is itself a sphere in $\mathbb{R}^D$ of radius $\sqrt{n(m-n)/m}$ centered at $\frac{n}{m}I_m$. A plane P and its orthogonal complement $P^\perp$ are represented by antipodal points on the large sphere.

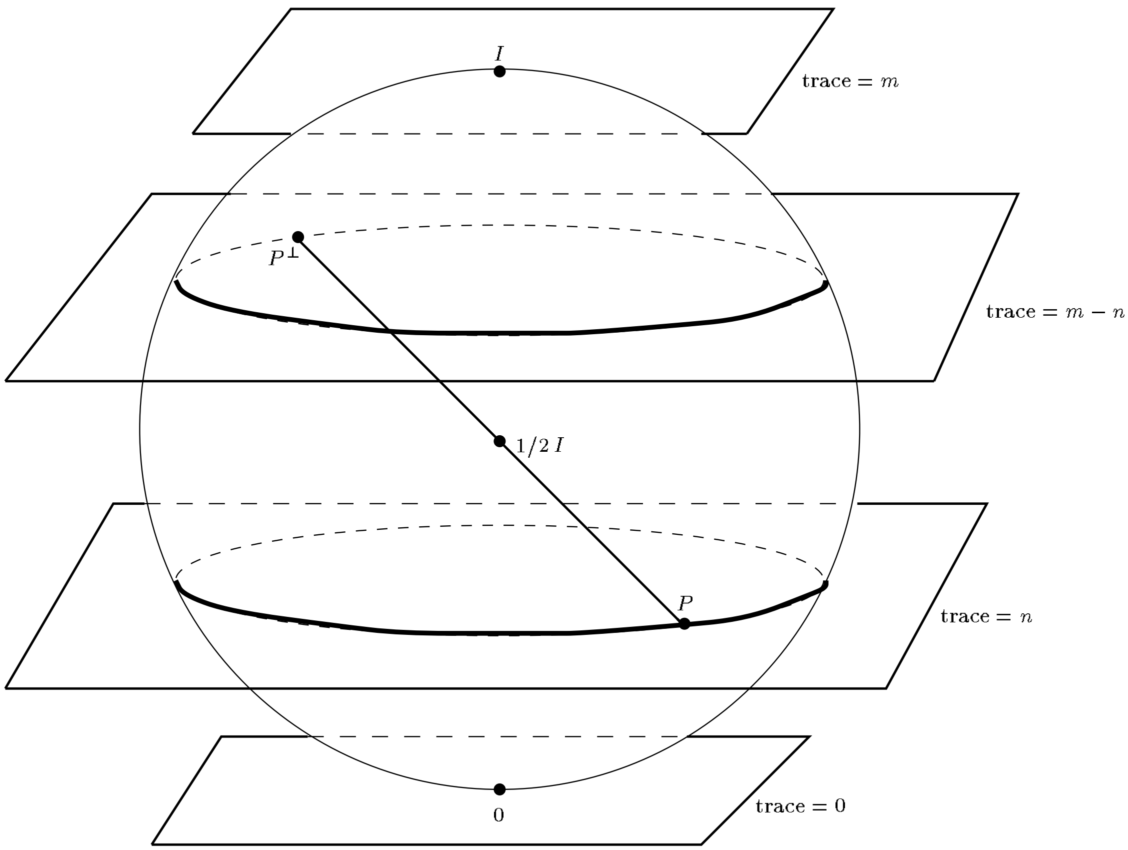

Figure 15: Embedding of $G(m,0),\ldots,G(m,m)$ into large sphere of radius $\sqrt{m}/2$ in Euclidean space of dimension $m(m+1)/2$. $G(m,n)$ lies on sphere of radius $\sqrt{n(m-n)/m}$ in $\mathbb{R}^D$, $D = (m-1)(m+2)/2$.

In contrast to this result, we briefly remark, without giving details, that there is no way to embed $G(m,n)$ into Euclidean space of any dimension so that the geodesic distance d_g on $G(m,n)$ is represented by Euclidean distance in that space. Of course the Plücker embedding, in which members of $G(m,n)$ are represented by points in projective space of dimension $\binom{m}{n}-1$, also does not give a way to realize either d_c or d_g as Euclidean distance. (Nor does the Nash embedding theorem [34].) Note also that the dimension of the Plücker embedding is in general much larger than the dimension of our embedding.

Since we have embedded $G(m, n)$ into a sphere of radius $\sqrt{n(m-n)/m}$ in $\mathbb{R}^D$, we can apply the Rankin bounds for spherical codes [35], and deduce:

Corollary. *(i) The simplex bound: for a packing of N planes in $G(m, n)$,*

$$d_c^2 \le \frac{n(m-n)}{m} \cdot \frac{N}{N-1} . \tag{5.5}$$

Equality requires $N \le D+1 = \binom{m+1}{2}$, and occurs if and only if the N points in $\mathbb{R}^D$ corresponding to the planes form a regular 'equatorial' simplex.

(ii) The orthoplex bound: for $N > \binom{m+1}{2}$,

$$d_c^2 \le \frac{n(m-n)}{m} . \tag{5.6}$$

Equality requires $N \le 2D = (m-1)(m+2)$, and occurs if the N points form a subset of the $2D$ vertices of a regular orthoplex. If $N = 2D$ this condition is also necessary.

Lemmons and Seidel ([31], Theorem 3.6) give a bound for equi-isoclinic packings in $G(m, n)$ which agrees with (5.5); of course our bound is more general. The case $n = 1$ of (5.5) is given in Theorem 3 of [38].

We were happy to obtain confirmation of (5.1). Since d_c^2 can never exceed n, we can also write

$$d_c^2 \le \min \left\{ n, \frac{n(m-n)}{m} \frac{N}{N-1} \right\} . \tag{5.7}$$

The Corollary allows us to establish the optimality of many of our packings. In the range $m \le 16$, $n \le 3$, $N \le 55$, there are over 750 cases which appear to meet (5.7) or (5.6), in the sense that the ratio of d_c^2 to the bound is greater than .9999999 . For $n = 2$, for instance, the following are the packings of N planes in $G(m, 2)$ for $m \le 10$ that appear to achieve either (5.7) (listed before the semicolon) or (5.6) (after):

$$\begin{aligned} m = 4 :&\quad 2-8, 10; 11-18 \\ m = 5 :&\quad 4-11; 16, 17 \\ m = 6 :&\quad 3-14; 22, 23 \\ m = 7 :&\quad 6-18; 29 \\ m = 8 :&\quad 4-21, 28; 37-44 \\ m = 9 :&\quad 6-25; \\ m = 10 :&\quad 5-28; \end{aligned}$$

However, this is not as meaningful as it at first seems. Consider the packings of N planes in $G(10,2)$. The ratios of d_c^2 to the simplex bound for $N = 21, \ldots, 33$, rounded to 14 decimal places, are as follows:

21	0.99999999999999
22	1.00000000000000
23	0.99999999999999
24	0.99999999999999
25	1.00000000000000
26	0.99999999999999
27	1.00000000000000
28	0.99999999921354
29	0.99999470290071
30	0.99998661084736
31	0.99961159256647
32	0.99909699728979
33	0.99854979246655

From this it seems very likely that there exists a 27-plane packing meeting the simplex bound, but certainly further investigation is needed to determine if there is a 28-plane simplex packing in the neighborhood of the computer's approximate solution. We have carried out such analyses in all the putative cases of equality for packing lines (see Sect. 6), for planes in $G(4,2)$ (as already discussed) and for 70 planes in $G(8,4)$ (see below). The other cases remain to be investigated.

In principle there is no difficulty in settling these questions. All the chordal distances between the planes are specified. So we could simply set up a (large) system of quartic equations with integer coefficients for the entries in the generator matrices, and ask if there is a real solution.

The chief difficulty in attempting to understand the computer-generated packings is that there are usually very large numbers of equally good solutions. The case of 18 planes in $G(4,2)$ is typical in this regard. There is one exceptionally pleasing solution (described in Section 4), in which only three different pairs of principal angles occur. However, it seems that there is a roughly 29-dimensional manifold of solutions, in which a typical solution, although still having $d_c^2 = 1$, has an apparently random set of principal angles. (Our investigation of this question is not yet completed.)

The case of subspaces of dimension n in $\mathbb{R}^{2n}$ or $\mathbb{R}^{2n+1}$ is especially interesting. The largest possible arrangements of planes that could achieve the two bounds are:

m	n	N(simplex)	N(orthoplex)
2	1	3 $\surd$	4 $\surd$
3	1	6 $\surd$	10
4	2	10 $\surd$	18 $\surd$
5	2	15	28
6	3	21	40
7	3	28 $\surd$	54
8	4	36	70 $\surd$
.	.	.	

(5.8)

Checks indicate that such a packing exists. It is known that the orthoplex bound cannot be achieved by 10 planes in $G(3,2)$, while the other cases are undecided. Our computer experiments strongly suggest that no set of 15 planes meets the simplex bound in $G(5,2)$.

On the other hand it is possible to find packings of 70 planes in $G(8,4)$ meeting the bound (5.6). With a considerable amount of effort (a long story, not told here), we determined several examples, of which the following is the most symmetrical. Let the coordinates be labeled $\infty, 0, 1, \ldots, 6$, and take two 4-planes generated by the vectors

$$\begin{array}{l} \{10000000,\ \ 01000000,\ \ 00100000,\ \ 00001000\} \\ \{11000000,\ \ 00101000,\ \ 00010001,\ \ 00000110\} \end{array}, \tag{5.9}$$

respectively. We obtain 70 planes from these by negating any even number of coordinates, and/or applying the permutations (0123456), $(\infty 0)(16)(23)(45)$ and (124)(365). The principal angles are $0, 0, \frac{\pi}{2}, \frac{\pi}{2}$; $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}$; or $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$, so $d_c^2 = 2$, $d_g^2 = \pi^2/4$ (this is not even a local optimum with respect to geodesic distance). The full group, which is transitive on the planes, is generated by the above operations and by the 8×8 Hadamard matrix

$$\frac{1}{\sqrt{8}} \begin{bmatrix} - & - & - & - & - & - & - & - \\ - & + & - & - & + & - & + & + \\ - & - & - & + & - & + & + & + \\ - & - & + & - & + & + & + & - \\ - & + & - & + & + & + & - & - \\ - & - & + & + & + & - & - & + \\ - & + & + & + & - & - & + & - \\ - & + & + & - & - & + & - & + \end{bmatrix}, \qquad \begin{array}{l} (- \text{ means } -1 \\ + \text{ means } +1) \end{array} .$$

It has shape $2^8 \mathcal{A}_8$ and order 5160960 ($\mathcal{A}_n$ denotes an alternating group of degree n).

A set of 28 planes in $G(7,3)$ meeting the simplex bound can be obtained as follows. Label the coordinates $0, \ldots, 6$, and let $v_r^{\pm}$ have components 1 at $2^r \pmod 7$, $\pm\sqrt{2}$ at $3.2^r \pmod 7$, and 0's elsewhere, for $r = 0, 1, 2$. Then the vectors $v_0^{\pm}$, $v_1^{\pm}$, $v_2^{\pm}$ (product of signs is even) span four planes, and the full set of 28 is found by cycling the seven coordinates.[4]

On the other hand, in spite of much effort, we have not been able to find a set of 40 planes in $G(6,3)$ that meet the bound. If this were possible, we would obtain $d_c^2 = 1.5$, whereas the best we have been able to achieve is 1.49977. We can get 1.5 with $N = 34$ planes, and 1.49977 with $35 \le N \le 40$, but only 1.4297 with 41 planes, which suggests that 1.5 might be possible with 40 planes. The discussion given earlier shows that this question could in principle be settled by seeing if a certain set of quadratic equations with integer coefficients has a real solution. One formulation leads to 500 quartic equations in 360 unknowns, so this approach seems hopeless at the present time.

It is not difficult to invent ways to construct sets of subspaces. Several examples are given below. However, many promising ideas have proved useless when confronted with the results found by our program. For this reason we have included two extensive tables of the best chordal distances that we have found. Table 3 gives d_c^2 for packings in $G(m,2)$, with $m \le 10$. This will serve as a standard against which readers can test their own constructions.

Table 3 about here

General constructions. The following are some promising general constructions.

(i) A skew-symmetric conference matrix[5] of order $4a$ yields a set of $4a$ unit vectors in $\mathbb{C}^{2a}$ with Hermitean inner products $\pm i/\sqrt{4a-1}$ ([14], Example 5.8), hence a set of $4a$ planes in $G(4a,2)$ with $d_c^2 = 4(2a-1)/(4a-1)$ that meet the simplex bound.

(ii) Use the planes defined by the n-faces of a regular m-dimensional polytope, or the Voronoi or Delaunay cells of a lattice, etc. The initial results from this idea have been disappointing. For example, the 96 two-dimensional faces of the 24-cell in $\mathbb{R}^4$ define 16 different planes, forming a packing with $d_c^2 = 8/9$, inferior to the best such packing (cf. Table 3).

(iii) Use the minimal vectors in a complex or quaternionic lattice to obtain packings in $G(2a,2)$ or $G(4a,4)$. For example, the 54 minimal vectors of the lattice E_6^*, regarded as a three-dimensional lattice over the Eisenstein integers ([9], p. 127), produce a packing of nine planes in $G(6,2)$ that meets the simplex bound.

[4]Note added April 1996. P. W. Shor and the third author have recently discovered that the packing of 70 planes in $G(8,4)$ can be generalized to a packing of m^2+m-2 planes in $G(m,m/2)$ meeting the orthoplex bound, whenever $m \ge 2$ is a power of 2; and that the packing of 28 planes in $G(7,3)$ can be generalized to a packing of $p(p+1)/2$ planes in $G(p,(p-1)/2)$ meeting the simplex bound, whenever p is a prime which is either 3 or congruent to -1 modulo 8. These constructions will be described elsewhere.

[5]For a list of the known orders of such matrices, see Table 7.1 of [40].

(iv) Restrict the search to generator matrices of 0's and 1's (as in (5.9)) or +1' and -1's, or even to rows that are blocks in some combinatorial design, or vectors in some error-correcting code (see the example in Section 6).

(v) Let C be an error-correcting code of length m over $GF(2^n)$, for example a Reed-Solomon code (cf. [33]). If we regard $GF(2^n)$ as a vector space of dimension n over $GF(2)$, each codeword yields an $n \times m$ matrix whose elements we may take to be +1's and -1's. After discarding those of rank less than n, and weeding out duplicates, we obtain a packing in $G(m, n)$. The hexacode ([9], p. 82), for example, a code of length 6 over $GF(4)$ containing 64 codewords, produces 28 distinct planes in $G(6, 2)$ with $d_c^2 = 3/4$. Unfortunately Table 4 shows that the record is 1.2973.

(vi) Choose a group G with an m-dimensional representation, and a subgroup H of index N with an n-dimensional irreducible representation. Find an n-dimensional subspace $V \subseteq \mathbb{R}^m$ invariant under H, and take its orbit under G. Many of the conference matrix constructions of line-packings described in Section 6 are of this type (using $G = L_2(q)$).

In some cases we have also used the optimizer to search for packings with a specified group. The following is a packing of 28 planes in $G(8, 2)$ meeting the simplex bound that is an abstraction of a configuration found by the computer when searching for packings in $\mathbb{R}^8$ invariant under the permutation $(0)(1, 2, 3, 4, 5, 6, 7)$. Let $R : \mathbb{C}^4 \to \mathbb{R}^8$ map (v_1, v_2, v_3, v_4) to $(\mathrm{Re}\, v_1, \mathrm{Im}\, v_1, \mathrm{Re}\, v_2, \ldots, \mathrm{Im}\, v_4)$, and let $\alpha = e^{2\pi i/7}$. The 28 planes are spanned by the following pairs of vectors:

$$\begin{array}{lll} R(0, \alpha^k, \alpha^{2k}, \alpha^{4k}), & R(0, i\alpha^k, i\alpha^{2k}, i\alpha^{4k}), & 0 \le k \le 6 \ , \\ R(1, 0, \alpha^{2k}, \alpha^{4k}), & R(i, 0, i\alpha^{2k}, -i\alpha^{4k}), & 0 \le k \le 6 \ , \\ R(1, \alpha^k, 0, \alpha^{4k}), & R(i, -i\alpha^k, 0, i\alpha^{4k}), & 0 \le k \le 6 \ , \\ R(1, \alpha^k, \alpha^{2k}, 0), & R(i, i\alpha^k, -i\alpha^{2k}, 0), & 0 \le k \le 6 \ . \end{array}$$

(The pattern of zeros and signs here suggests the Tetracode [9], p. 81.)

An application: apportioning randomness. The following is a potential application of packings in $G(m, n)$ for use in apportioning randomness, in the sense of producing large numbers of approximately random points from a few genuinely random numbers. We illustrate using the example of 70 planes in $G(8, 4)$ described above. Let $A_1, \ldots, A_{70}$ be generator matrices for them. Suppose an exclusive resort wishes to distribute random points in $\mathbb{R}^4$ to its 70 guests, perhaps for use as garage door keys. Let $x = (x_1, \ldots, x_8)$ be a vector of eight independent Gaussian random variates. Then the hotel would assign $y_i = A_i x^{tr}$ to its i-th guest. By maximizing the chordal distance between the planes we have minimized the correlation between the y_i.

6. Packing lines in higher dimensions

Table 4 shows the maximal angular separation found for packings of $N \leq 50$ lines in $G(m, 1)$ for $m \leq 9$.

The following packings of N lines in $G(m, 1)$ for $m \leq 10$ achieve either the simplex or orthoplex bounds: $m = 3: N = 3, 4, 6, 7$; $m = 4: N = 4, 5, 11, 12$; $m = 5: N = 5, 6, 10, 16$; $m = 6: N = 6, 7, 16, 22$; $m = 7: N = 7, 8, 14, 28$; $m = 8: N = 8, 9$; $m = 9: N = 9, 10, 18, 46 - 48$, $m = 10: N = 10, 11, 16$. Most of these are described below.

As mentioned in Section 3, the best packing of 5 lines in $G(3, 1)$ is a subset of the best packing of 6 lines. Table 4 shows similar phenomena in higher dimensions. For example, the putatively best packing of 48 lines in $G(9, 1)$ is so good that we cannot do better even when up through 8 lines are omitted from it.

In the rest of this section we discuss some of the entries in Table 4 that have the largest symmetry groups.

Table 4 about here

Constructions from lattices. Let $P_1, \ldots, P_a$ be a set of mutually touching spheres in a d-dimensional lattice packings. If there are $2N$ further spheres in the packing, each of which touches all of $P_1, \ldots, P_a$, their centers form a $(d - a + 1)$-dimensional antipodal spherical code with angular separation $\mathrm{arcsec}(a + 1)$ ([9], p. 340, Theorem 1).

In particular, the entries labeled "1" in Table 4 are obtained from the centers of spheres that touch one sphere in the lattices D_4, D_5, E_6 and E_7. The entries labeled "2" are obtained from spheres that touch two spheres in the lattices D_4, D_5, E_6, E_7, E_8, Λ_9 and the nonlattice packing P_{10b} (cf. [9], Chap. 1, Table 1.2). We remark the set of 28 lines in $G(7, 1)$ with angle $\arccos(1/3)$ obtained in this way from E_8 is known to be unique: see Chapter 14, Theorem 12 of [9]. This configuration of lines is derived in a different way in [32].

It is interesting to compare Table 4 with the table of maximal sets of equiangular lines given in Lemmens and Seidel [30] and Seidel [41]. Some arrangements appear in both tables, for example the set of 28 lines in $G(7, 1)$ just mentioned. On the other hand the difference between the tables can be seen in dimension 8. The maximal set of equiangular lines that exists in $G(8,1)$ has size 28, with angle arcsec 3 ([30], Theorem 4.6). However, Table 4 shows that there is a set of 28 lines (not equiangular) in $G(8,1)$ with minimal angle arcsec 3.000511... . In view of the Lemmons-Seidel result (which uses the fact that in an m-dimensional equiangular arrangement of N lines with $N \geq 2m$ the secant must be an odd integer) our set of 28 lines cannot be perturbed to give an equiangular set without decreasing the minimal angle.

From conference matrices. It follows from Theorem 6.3 of [32] that if a symmetric conference matrix of order $q+1 \equiv 2 \pmod 4$ exists then there is an arrangement of $q+1$ equiangular lines in $\mathbb{R}^m$, $m = (q+1)/2$, with $d_c^2 = (q-1)/q$, meeting the simplex bound. The corresponding entries are labeled "3" in the table. Our program was able to find these packings for every prime power q of this form below 100 except for 49 and 81.

From diplo-simplices. The entries labeled "4" are obtained by using all the vectors of shape $\pm c(n^1, (-1)^n)$, where $c = 1/\sqrt{n(n+1)}$. In the notation of [9], Chap. 4, these are the minimal vectors in the translates of the root lattice A_n by the glue vectors [1] and [−1]. They form the vertices of a diplo-simplex [8], and have automorphism group $2 \times \mathcal{A}_{n+1}$. These packings also meet the simplex bound.

From codes. Let C be a binary code of length m, size M and minimal distance d which is closed under complementation. Writing the codewords as vectors of ± 1's, we obtain a packing of $M/2$ lines in $G(m,1)$ with $d_c^2 = 4d(m-d)/m^2$. For example, a shortened Hamming code of length 10 with $M = 32$, $d = 4$ gives a packing of 16 lines in $G(10,1)$ meeting the simplex bound. This construction provides a rich supply of good packings. The Nordstrom-Robinson code [17], [21], for example, yields a packing of 128 lines in $G(16,1)$ with $d_c^2 = 15/16$.

40 lines in $G(4,1)$. The 80 antipodal points are $(\xi^{\mu+1/2}, 0)$, $(0, \xi^{\nu+1/2})$, $(a\xi^{2\mu}, b\xi^{2\nu+1})$, $(a\xi^{2\mu+1}, b\xi^{2\nu})$, $(b\xi^{2\mu}, a\xi^{2\nu})$, $(b\xi^{2\mu+1}, a\xi^{2\nu+1})$, where $\xi = e^{2\pi i/8}$, $\mu, \nu = 0, \ldots, 3$, $a = 2^{-1/4}$, $b = \sqrt{1-a^2}$. The group of these 80 points is the group $\frac{1}{2}(\mathcal{D}_{16} + \mathcal{D}_{16}) \cdot 2$ of order 256 (see [10]). Here $\mathcal{D}_m$ denotes a dihedral group of order m.

22 lines in $G(6,1)$. We use the vertices of a hemi-cube (an alternative way to describe the 16-line packing) together with the six coordinate axes.

63 and 64 lines in $G(7,1)$. The putatively best packing of 63 lines in $G(7,1)$ (just beyond the range of Table 4) has angular separation 60°, and is formed from the 126 minimal vectors of the lattice E_7. The automorphism group of this set of 126 points is the Weyl group $\mathcal{W}(E_7)$, of order $2^{10}.3^4.5.7$ (cf. [9], Chap. 4, §8.3).

The best packing found of 64 lines in $G(7,1)$ also has an unusually large group, of order $2|\mathcal{W}(E_6)| = 2^8.3^4.5$. This packing can be obtained as follows. The largest gaps between the packing of 63 lines occur in the directions of the minimal vectors of the dual lattice E_7^*. Let u be such a vector, with $u = {}^1\!/_2\, v_6$, $v_6 \in E_7$, $v_6 \cdot v_6 = 6$. Taking a coordinate frame in which the last coordinate is in the v_6 direction, we obtain the minimal vectors of E_7 in the form $(w, 0)$, $w \in E_6$, $w \cdot w = 2$ (72 vectors), and $(x, \pm\sqrt{2/3})$, $x \in E_6^*$, $x \cdot x = 4/3$ (54 vectors). We now adjoin $\pm\sqrt{6}$ and rescale by multiplying the last coordinate by $\sqrt{3/4}$, obtaining our final configuration of vectors $(0, \pm 3/\sqrt{2})$, $(w, 0)$, $(x, \pm 1/\sqrt{2})$, which is isomorphic to what the computer found. The rescaling has compressed the E_7 lattice in the v_6 direction until the angle between v_6 and the $(x, 1/\sqrt{2})$ vectors is the same as the minimal angle between the $(x, 1/\sqrt{2})$ and $(w, 0)$ vectors. This angle, the minimal angle in the packing, is $\arccos(\sqrt{3/11}) = 58.5178°$.

36 lines in $G(8,1)$. The 72 antipodal points consist of all permutations of the two vectors $c(7^2, -2^7)$ and $c(-7^2, 2^6)$, where $c = 1/\sqrt{126}$. These are the minimal vectors in the translates of the A_8 lattice by the glue vectors [2] and [−2].

The configuration of 16 lines in $G(5,1)$ is similarly obtained from the translates of A_5 by [1], [3] and [5].

39 lines in $G(12,1)$. Take the 13 lines of a projective plane of order 3, described by vectors of four 1's and nine 0's, and replace two of the 1's by −1's in all possible ways, obtaining 78 vectors in $\mathbb{R}^{12}$ with angle arcsec 4. The group is $2 \times L_3(3)$, of order $2^5 3^3 13$.

If instead we replace any *odd* number of the 1's by −1's, we obtain a putatively optimal packing of 52 lines in $G(13,1)$, with angle arcsec 4 and group $2 \times L_3(3).2$ of order $2^6 3^3 13$.

As in all these examples, we were amazed that the program was able to discover such beautiful configurations.

Acknowledgements

We are grateful to Daniel Asimov, Andreas Buja, Dianne Cook, Simon Kochen, Warren Smith, and especially Colin Mallows for helpful discussions, and to David Applegate for help in solving the matching problem described in Section 4.

Table 3: Best packings found of $N \leq 50$ planes in $G(m,2)$, $m \leq 10$. The entry gives d_c^2.

$N \setminus m$	4	5	6	7	8	9	10
3	1.5000	1.7500	2.0000	2.0000	2.0000	2.0000	2.0000
4	1.3333	1.6000	1.7778	1.8889	2.0000	2.0000	2.0000
5	1.2500	1.5000	1.6667	1.7854	1.8750	1.9375	2.0000
6	1.2000	1.4400	1.6000	1.7143	1.8000	1.8667	1.9200
7	1.1667	1.4000	1.5556	1.6667	1.7500	1.8148	1.8667
8	1.1429	1.3714	1.5238	1.6327	1.7143	1.7778	1.8286
9	1.1231	1.3500	1.5000	1.6071	1.6875	1.7500	1.8000
10	1.1111	1.3333	1.4815	1.5873	1.6667	1.7284	1.7778
11	1.0000	1.3200	1.4667	1.5714	1.6500	1.7111	1.7600
12	1.0000	1.3064	1.4545	1.5584	1.6364	1.6970	1.7455
13	1.0000	1.2942	1.4444	1.5476	1.6250	1.6852	1.7333
14	1.0000	1.2790	1.4359	1.5385	1.6154	1.6752	1.7231
15	1.0000	1.2707	1.4286	1.5306	1.6071	1.6667	1.7143
16	1.0000	1.2000	1.4210	1.5238	1.6000	1.6593	1.7067
17	1.0000	1.2000	1.4127	1.5179	1.5937	1.6528	1.7000
18	1.0000	1.1909	1.4048	1.5126	1.5882	1.6471	1.6941
19	0.9091	1.1761	1.3948	1.5078	1.5833	1.6420	1.6889
20	0.9091	1.1619	1.3888	1.5026	1.5789	1.6374	1.6842
21	0.8684	1.1543	1.3821	1.4987	1.5750	1.6333	1.6800
22	0.8629	1.1419	1.3333	1.4912	1.5714	1.6296	1.6762
23	0.8451	1.1332	1.3333	1.4859	1.5680	1.6263	1.6727
24	0.8372	1.1251	1.3326	1.4790	1.5638	1.6232	1.6696
25	0.8275	1.1178	1.3229	1.4725	1.5594	1.6204	1.6667
26	0.8144	1.1113	1.3151	1.4666	1.5556	1.6177	1.6640
27	0.8056	1.1045	1.3071	1.4606	1.5556	1.6154	1.6615
28	0.8005	1.0989	1.2987	1.4531	1.5556	1.6118	1.6593
29	0.7889	1.0937	1.2887	1.4286	1.5455	1.6083	1.6571
30	0.7809	1.0875	1.2804	1.4234	1.5398	1.6049	1.6552
31	0.7760	1.0822	1.2675	1.4167	1.5342	1.6011	1.6527
32	0.7691	1.0766	1.2588	1.4106	1.5304	1.5978	1.6501
33	0.7592	1.0722	1.2526	1.4038	1.5244	1.5935	1.6476
34	0.7549	1.0671	1.2447	1.3978	1.5216	1.5911	1.6448
35	0.7489	1.0640	1.2430	1.3915	1.5158	1.5893	1.6431
36	0.7477	1.0596	1.2345	1.3843	1.5086	1.5885	1.6414
37	0.7286	1.0519	1.2272	1.3784	1.5000	1.5816	1.6364
38	0.7198	1.0462	1.2239	1.3722	1.5000	1.5768	1.6334
39	0.7095	1.0404	1.2206	1.3663	1.5000	1.5730	1.6305
40	0.7066	1.0355	1.2149	1.3606	1.5000	1.5695	1.6289
41	0.6992	1.0302	1.2115	1.3575	1.5000	1.5665	1.6242
42	0.6948	1.0256	1.2079	1.3550	1.5000	1.5639	1.6213
43	0.6844	1.0207	1.2037	1.3449	1.5000	1.5616	1.6192
44	0.6831	1.0159	1.2007	1.3407	1.5000	1.5588	1.6174
45	0.6809	1.0122	1.1969	1.3354	1.4839	1.5570	1.6160
46	0.6793	1.0078	1.1941	1.3321	1.4821	1.5480	1.6099
47	0.6732	1.0042	1.1924	1.3265	1.4574	1.5412	1.6072
48	0.6667	1.0001	1.1907	1.3232	1.4490	1.5353	1.6045
49	0.6667	0.9960	1.1873	1.3209	1.4430	1.5304	1.6019
50	0.6667	0.9910	1.1841	1.3150	1.4370	1.5257	1.6003

Table 4: Maximal angular separation found for $N \le 50$ lines in $G(m,1)$, $m \le 9$.

$N \setminus m$	3	4	5	6	7	8	9
3	90.0000	90.0000	90.0000	90.0000	90.0000	90.0000	90.0000
4	70.5288[2]	90.0000	90.0000	90.0000	90.0000	90.0000	90.0000
5	63.4349[6]	75.5225[4]	90.0000	90.0000	90.0000	90.0000	90.0000
6	63.4349[3]	70.5288[2]	78.4630[4]	90.0000	90.0000	90.0000	90.0000
7	54.7356[6]	67.0213	73.3689	80.4059[4]	90.0000	90.0000	90.0000
8	49.6399[6]	65.5302	70.8039	76.0578	81.7868[4]	90.0000	90.0000
9	47.9821[6]	64.2619	70.5288	73.8437	78.4630	82.8192[4]	90.0000
10	46.6746[6]	64.2619	70.5288[2]	73.6935	76.3454	79.4704	83.6206[4]
11	44.4031	60.0000	67.2543	71.5651	75.0179	77.8695	80.6204
12	41.8820[6]	60.0000[1]	67.0213	71.5651	74.1734	76.6050	79.4704
13	39.8131	55.4646	65.7319	70.5288	73.8979	76.1645	77.9422
14	38.6824	53.8376	65.7241	70.5288	73.8979[3]	75.0349	77.2382
15	38.1349[6]	52.5016	65.5302	70.5288	71.5678	74.3318	76.5006
16	37.3774[6]	51.8273	63.4349[5]	70.5288[2]	70.9861	74.1005	75.9638
17	35.2353	50.8870	61.2551	68.1088	70.5926	73.1371	75.9638
18	34.4088	50.4577	61.0531	67.3744	70.5527	72.7464	75.9638[3]
19	33.2115	49.7106	60.0000	67.3700	70.5288	72.0756	74.4577
20	32.7071	49.2329	60.0000[6]	67.0996	70.5288	71.6706	74.2278
21	32.2161	48.5479	57.2025	67.0213	70.5288	71.3521	73.7518
22	31.8963	47.7596	56.3558	65.9052[5]	70.5288	71.0983	73.1894
23	30.5062	46.5104	55.5881	63.6744	70.5288	70.7720	72.7488
24	30.1628	46.0478	55.2279	63.6122	70.5288	70.6027	72.6547
25	29.2486	44.9471	54.8891	62.4240	70.5288	70.5490	72.3124
26	28.7126	44.3536	54.2116	61.7377	70.5288	70.5432	72.1763
27	28.2495	43.5530	53.5402	61.4053	70.5288	70.5392	71.6650
28	27.8473	43.1566	53.2602	60.5276	70.5288[2]	70.5322	71.5794
29	27.5244	42.6675	53.0180	60.1344	66.7780	70.5288	71.5175
30	26.9983	42.2651	52.7812	60.0213	65.7563	70.5288	71.5175
31	26.4987	42.0188	52.4120	60.0000	65.1991	70.5288	70.8508
32	25.9497	41.9554	52.3389	60.0000	64.7219	70.5288	70.7437
33	25.5748	41.4577	52.2465	60.0000	64.6231	69.3203	70.6940
34	25.2567	40.9427	51.8537	60.0000	64.6231	69.1688	70.6512
35	24.8702	40.7337	51.8273	60.0000	64.6231	69.0752	70.6337
36	24.5758	40.6325	51.8273	60.0000[1]	64.6231	69.0752[5]	70.5864
37	24.2859	40.4486	51.8273	57.6885	62.3797	67.7827	70.5695
38	24.0886	40.4419	50.3677	57.1057	62.1435	67.3835	70.5571
39	23.8433	39.6797	50.0611	56.8357	61.7057	67.1387	70.5443
40	23.3293	39.0236[5]	49.5978	56.0495	61.3792	66.3815	70.5288
41	22.9915	38.5346	49.2600	55.8202	61.1630	65.9282	70.5288
42	22.7075	38.3094	48.6946	55.6160	60.8232	65.8166	70.5288
43	22.5383	37.7833	48.4030	55.3981	60.5193	65.2885	70.5288
44	22.2012	37.3474	48.0955	55.1259	60.3623	65.2422	70.5288
45	22.0481	37.1198	47.7723	54.9980	60.2282	64.7476	70.5288
46	21.8426	36.9997	47.3753	54.9858	60.1101	64.4007	70.5288
47	21.6609	36.5952	47.0323	54.7356	60.0433	64.1542	70.5288
48	21.4663	36.3585	46.7105	54.5940	60.0116	63.8846	70.5288[2]
49	21.1610	36.1369	46.4345	54.5031	60.0000	63.4849	68.0498
50	20.8922	36.0754	46.1609	54.3191	60.0000[1]	63.1527	67.7426

Key. 1,2: from sphere packings; 3: from conference matrices; 4: diplo-simplex; 5: described below; 6: described in Section 3.

References

[1] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, Wiley, NY, 2nd ed., 1984, Chap. 12.

[2] D. Applegate, R. Bixby, V. Chvátal and W. Cook, Finding cuts in the TSP (a preliminary report), *DIMACS Technical Report 95-05*, DIMACS Center, Rutgers University, New Brunswick, NJ, March, 1995.

[3] D. Asimov, The Grand Tour — a tool for viewing multidimensional data, *SIAM J. Sci. Stat. Comput.*, **6** (1985), 128–143.

[4] A. Björck and G. H. Golub, Numerical methods for computing angles between linear subspaces, *Math. Comp.* **27** (1973), 579–594.

[5] A. Buja and D. Asimov, Grand Tour methods: an outline, in *Computer Science and Statistics: The Interface*, ed. D. M. Allen, Elsevier, 1986, pp. 63–67.

[6] W.-L. Chow, On the geometry of algebraic homogeneous spaces, *Annals Math.*, **50** (1949), 32–67.

[7] J. H. Conway and S. Kochen, Geometry and quantum logic, in preparation.

[8] J. H. Conway and N. J. A. Sloane, The cell structures of certain lattices, in *Miscellanea mathematica*, P. Hilton, F. Hirzebruch and R. Remmert, Eds., Springer-Verlag, NY, 1991, pp. 71–107.

[9] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, NY, 2nd ed., 1993.

[10] J. H. Conway and N. J. A. Sloane, The four-dimensional point groups — a revised enumeration, in preparation.

[11] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* **56** (1950), 413–455.

[12] H. S. M. Coxeter, Polyhedral numbers, in R. S. Cohen et al., editors, *For Dirk Struik*, Reidel, Dordrecht, 1974, pp. 25–35.

[13] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, NY, 4th ed., 1980.

[14] P. Delsarte, J. M. Goethals and J. J. Seidel, Bounds for systems of lines, and Jacobi polynomials, *Philips Research Reports*, **30** (1975), 91–105.

[15] P. Du Val, *Homographies, Quaternions and Rotations*, Oxford Univ. Press, 1964.

[16] L. Fejes Tóth, Distribution of points in the elliptic plane, *Acta Math. Acad. Sci. Hungar.*, **16** (1965), 437–440.

[17] G. D. Forney, Jr., N. J. A. Sloane, and M. D. Trott, The Nordstrom-Robinson Code is the Binary Image of the Octacode, *Coding and Quantization: DIMACS/IEEE Workshop October 19–21, 1992*, R. Calderbank, G. D. Forney, Jr., and N. Moayeri, Eds., Amer. Math. Soc., 1993, pp. 19–26.

[18] H. Gluck, D. MacKenzie and F. Morgan, Volume-minimizing cycles in Grassmann manifolds, *Duke Math J.*, **79** (1995), 335–404.

[19] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins Univ. Press, 2nd ed., 1989.

[20] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, NY, 1978, esp. Chap. 1, §5.

[21] A. R. Hammons, Jr., P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Solé, The $\mathbb{Z}_4$-Linearity of Kerdock, Preparata, Goethals and Related Codes, *IEEE Trans. Information Theory*, **40** (1994), pp. 301–319.

[22] R. H. Hardin and N. J. A. Sloane, *Operating Manual for Gosset: A General-Purpose Program for Constructing Experimental Designs (Second Edition)*. Statistics Research Report No. 106, Bell Laboratories, Murray Hill, NJ, October, 1992. Also DIMACS Technical Report 93–51, DIMACS Center, Rutgers University, New Brunswick, NJ, August 1993.

[23] R. H. Hardin and N. J. A. Sloane, A New Approach to the Construction of Optimal Designs, *J. Statistical Planning and Inference*, **37** (1993), 339–369.

[24] R. H. Hardin, N. J. A. Sloane and W. D. Smith, *Tables of Spherical Codes in Dimensions 3, 4, 5*, Netlib archive, `http://netlib.att.com/math/sloane/packings`, 1993–1995. This table is based on our own calculations, with additional packings provided by D. A. Kottwitz and J. Buddenhagen, and is believed to contain the best packings known with $N \leq 130$ points in dimensions 3, 4 and 5.

[25] R. H. Hardin, N. J. A. Sloane, and W. D. Smith, *Spherical Codes*, book in preparation.

[26] R. Hooke and T. A. Jeeves, 'Direct search' solution of numerical and statistical problems, *J. Assoc. Comput. Math.*, **8** (1961), 212–229.

[27] R. W. H. T. Hudson, *Kummer's Quartic Surface*, Cambridge Univ. Press, 1905.

[28] A. T. James and A. G. Constantine, Generalized Jacobi polynomials as spherical functions of the Grassmann manifold, *Proc. London Math. Soc.*, **29** (1974), 174–192.

[29] K. Leichtweiss, Zur Riemannschen Geometrie in Grassmannschen Mannigfaltigkeiten, *Math. Zeit.*, **76** (1961), 334–366.

[30] P. W. H. Lemmens and J. J. Seidel, Equiangular lines, *J. Algebra*, **24** (1973), 494–512.

[31] P. W. H. Lemmens and J. J. Seidel, Equi-isoclinic subspaces of Euclidean space, Konink. Ned. Akad. Wet., **A 76** (1973), 98–107.

[32] J. H. van Lint and J. J. Seidel, Equilateral point sets in elliptic geometry, *Proc. Kon. Nederl. Akad. Wet. Series A*, **69** (1966), 335–348.

[33] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, Elsevier, Amsterdam, 1977.

[34] J. Nash, The imbedding problem for Riemannian manifolds, *Annals Math.*, **63** (1956), 20–63.

[35] R. A. Rankin, The closest packing of spherical caps in n dimensions, *Proc. Glasgow Math. Assoc.*, **2** (1955), 139–144.

[36] M. Rosenfeld, Problem No. 10293, *Amer. Math. Monthly*, **100** (1993), 291.

[37] M. Rosenfeld, How wide can you spread your chopsticks? *Congressus Numerantium*, **102** (1994), 29–31.

[38] M. Rosenfeld, In praise of the Gram matrix, preprint.

[39] J. Rosenman, The rotation group and cancer treatment, Electronic message posted to `sci.math` newsgroup on the Internet, Oct. 29, 1992.

[40] J. Seberry and M. Yamada, Hadamard matrices, sequences, and block designs, in *Contemporary Design Theory*, ed. J. H. Dinitz and D. R. Stinson, Wiley, 1992, pp. 431–560.

[41] J. J. Seidel, Discrete non-Euclidean geometry, in *Handbook of Incidence Geometry*, ed. F. Buekenhout, North-Holland, Amsterdam, 1994, Chap. 21.

[42] C. Stephanos, Sur les systèmes desmiques de trois tétraèdres, *Bull. Sci. Math.* **3** (1879), 424–456.

[43] D. F. Swayne, D. Cook and A. Buja, XGobi: interactive dynamic graphics in the X window system with a link to S, in *ASA Proceedings Section on Statistical Graphics*, Amer. Stat. Assoc., Alexandria, VA, 1991, pp. 1–8.

[44] Y.-C. Wong, *Isoclinic n-Planes in Euclidean $2n$-Space, Clifford Parallels in Elliptic $(2n-1)$-Space, and the Hurwitz Matrix Equations*, Memoirs Amer. Math. Soc., Vol. 41, 1961.

[45] Y.-C. Wong, Differential geometry of Grassmann manifolds, *Proc. Nat. Acad. Sci. USA*, **57** (1967), 589–594.

[46] C. Zanella, Embeddings of Grassmann spaces, *J. Geometry*, **52** (1995), 193–201.