The Primary Pretenders

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July 20, 2002

Perhaps the most famous theorem in number theory is Fermat's theorem. Not Fermat's Last Theorem, of course, because that's now old hat, but Fermat's Little Theorem:

If p is a prime, and b is a positive integer prime to p, then $b^{p-1} \equiv 1 \pmod{p}$,

which we prefer to write in the simpler form

$$b^p \equiv b \pmod{p}.$$

If the converse of the theorem were true, then number theory would be a lot simpler than it is, but fortunately that is not the case. Counterexamples to the converse of the first (and, very occasionally, the second) form of Fermat's theorem are called **pseudoprimes**. A well-known example is $341 = 11 \times 31$, which is a pseudoprime to base 2:

$$2^{340} \equiv 1 \pmod{341}$$

The literature on pseudoprimes is extensive; for an introduction see section **A12** of the second author's *Unsolved Problems in Number Theory*, 2nd edition, Springer, 1994. D.H. Lehmer found the even pseudoprime $161038 = 2 \cdot 73 \cdot 1103$ and N.G.W.H. Beeger showed that there were infinitely many.

The **Carmichael numbers**, such as $561 = 3 \times 11 \times 17$, are counterexamples to the second form of Fermat's theorem to *any* base:

$$b^{561} \equiv b \pmod{561}\ , \quad b = 1, 2, \ldots\ .$$

The second form of the theorem admits a much wider class of counterexamples than the first, and to distinguish them from the pseudoprimes we will call any composite number q such that $b^q \equiv b \pmod{q}$ a **prime pretender** to base b.

We investigate q_b, the least prime pretender, or **primary pretender**, for the base b.

We will see that there are only 132 distinct primary pretenders, and that q_b is a periodic function of b whose period is the 122-digit number

19 5685843334 6007258724 5340037736 2789820172 1382933760 4336734362-
2947386477 7739548319 6097971852 9992599213 2923650684 2360439300

What is this number? Well, it's $p!_{59}p!_9$, where $p!_k$ is the product of the first k primes, $p_1p_2\cdots p_k$. And where do p_{59} and p_9 come from? $p_{59} = 277$ is the largest possible prime factor, and $p_9 = 23$ is the largest possible repeated prime factor, of a composite number less than the Carmichael number 561.

For what bases is 4 a prime pretender? If $b \equiv 0, 1, 2, 3 \pmod 4$, then $b^4 \equiv 0, 1, 0, 1$, so 4 is a prime pretender just for $b \equiv 0, 1 \pmod 4$.

The similar calculations mod 6 and 8 show that 6 is a prime pretender for bases $\equiv 0$ or $1 \pmod 3$ and that 8 is a prime pretender for bases $\equiv 0$ or $1 \pmod 8$. It follows that every number for which 8 is a prime pretender also has 4 as a prime pretender, so that 8 can never be the *primary* pretender. The calculations mod 9 show that 9 is a prime pretender for bases $\equiv 0, 1$ or $8 \pmod 9$, which may also be described as the square roots of 0 or $1 \pmod 9$.

These results can be recorded by saying that for $q = 4$ and 9,

"q is a prime pretender just for the bases that are kth roots of 0 or 1 (mod m)"

for a certain k and m. (It will turn out that such an assertion holds for all the primary pretenders — see Table 3.) They imply that we know the *primary* pretender q_b for all but the four residue classes 2, 11, 14, 23 (mod 36):

$b \equiv$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
$q_b =$	4	4	?	6	4	4	6	6	4	4	6	?	4	4	?	6	4	4	6	6	4	4	6	?	4	4	9	6	4	4	6	6	4	4	6	9

The values of q_b up to 21 for the residue classes mod 1260 missing from the last display are given in Table 1. In fact $q_b \geq 22$ for just the 32 residue classes mod 1260 indicated by ? in Table 1.

The number of distinct values of q_b is bounded, since the Carmichael number 561 will always serve if no smaller exponent has been found. The other numbers which occur

Table 1: $q_b = 10, 14, 15, 21$ for just 108 residue classes mod 1260.

$b =$	2	11	14	23	38	47	50	59	74	83	86	95	110	119	122	131	146	155	158	167
+0	?	10	14	?	?	?	10	15	15	21	10	10	10	14	?	10	10	10	?	21
+180	14	10	15	14	14	?	10	14	15	?	10	10	10	15	14	10	10	10	?	?
+360	?	10	15	?	21	14	10	15	14	?	10	10	10	15	21	10	10	10	14	?
+540	?	10	14	?	?	21	10	15	15	14	10	10	10	14	?	10	10	10	?	14
+720	14	10	15	14	?	?	10	15	15	?	10	10	10	15	?	10	10	10	?	?
+900	21	10	15	21	14	?	10	14	14	?	10	10	10	15	14	10	10	10	14	?
+1080	?	10	15	?	?	14	10	15	15	14	10	10	10	15	?	10	10	10	21	14

are products of just two prime factors: twice the primes from 2 to 277; thrice the primes from 3 to 181; five times those primes which are $\equiv 1 \pmod 4$ from 5 to 109; seven times those primes which are $\equiv 1 \pmod 3$ from 7 to 79; eleven times 11, 31 & 41; thirteen times 13 & 37; and the squares of 17, 19 & 23.

Computer calculations of the numbers in the missing residue classes for values of b up to 50000 appear in Table 2; the numbers at the left show the multiples of 1260 to be added. The programs used to calculate Tables 2 and 3 were straightforward, essentially using brute force.

Our final table, Table 3, shows how long it takes before any particular value of q_b appears; it can be summarized as follows. The value of q_b is

$$\begin{array}{lrlcl} & 4 & \text{if } b \equiv & 0,1 & \pmod 4 \\ \text{else} & 6 & \text{if } b \equiv & 0,1 & \pmod 3 \\ \text{else} & 9 & \text{if } b \equiv & 8 & \pmod 9 \\ \text{else} & \ldots & \ldots & \ldots & \ldots \\ \text{else} & 561 & \text{if } b \equiv & 0 & \pmod 1 \end{array}$$

where the various statements can all be put into the form

"else q if b is a kth root of 0 or 1 (mod m)"

for appropriate values of q, k and m. The table also gives the **first base**, that is the least b for which $q_b = q$, and the **rarity** r of q, meaning that q is the primary pretender for 1 in every r bases. For example

$$25 \qquad 4\text{th}(25) \qquad 443 \qquad 240.62$$

Table 2: $q_b \geq 22$ for 32 residue classes mod 1260, $2 \leq b \leq 51602$.

$b =$	2	23	38	47	122	158	227	263	338	347	362	383	443	527	542	563	578	662	698	758	767	803	842	878	887	947	983	1067	1082	1103	1118	1202
0	341	22	38	46	22	158	49	33	26	87	33	382	25	33	91	91	34	39	34	33	26	22	58	259	91	22	65	22	25	38	25	169
1	26	91	22	25	25	25	65	185	34	22	91	25	26	38	34	57	22	34	22	25	39	91	22	49	38	25	25	26	33	34	39	46
2	26	25	91	34	38	26	82	22	113	94	22	33	39	22	145	46	38	25	25	22	38	22	49	33	25	51	34	22	26	91	34	51
3	25	49	22	33	49	22	25	25	25	85	38	46	33	25	33	25	91	91	22	26	91	58	69	34	26	34	22	74	22	33	62	25
4	121	122	49	85	26	46	46	22	33	65	22	22	91	22	25	26	25	49	38	22	25	25	25	22	39	82	38	25	39	25	85	49
5	33	51	133	22	26	22	26	34	65	34	133	26	22	145	22	33	26	33	91	39	57	65	65	74	85	133	22	58	22	22	25	22
6	38	34	51	25	25	25	26	91	22	25	34	22	65	26	91	62	26	365	46	25	22	51	62	22	33	25	25	38	49	57	33	26
7	22	25	34	22	33	65	39	38	38	85	25	39	22	26	22	22	39	22	25	91	51	34	91	26	25	33	26	133	91	22	38	22
8	25	561	25	26	57	58	22	25	22	26	46	327	34	25	26	25	33	46	26	91	22	341	33	39	22	74	26	142	65	91	22	25
9	22	22	62	39	22	121	86	82	51	26	141	38	65	34	25	22	25	22	26	46	25	25	25	25	85	22	39	25	123	25	91	85
10	65	26	33	51	49	49	22	38	34	22	26	91	25	91	34	49	22	38	33	38	82	26	22	46	22	26	34	51	25	26	22	74
11	58	22	26	25	22	25	65	33	62	25	26	25	91	33	38	205	65	26	58	25	69	22	39	51	65	22	25	22	51	26	34	34
12	49	25	22	58	145	33	85	91	26	22	25	46	49	65	49	65	22	25	22	34	26	38	22	34	25	39	33	57	33	39	26	38
13	25	91	25	65	58	46	25	22	25	49	22	33	26	22	91	25	62	39	91	22	26	22	34	33	49	49	65	22	38	93	26	25
14	26	34	22	33	91	22	34	49	39	34	49	65	26	62	25	38	25	451	22	85	25	25	25	25	118	91	22	25	22	25	39	33
15	26	57	34	169	46	26	62	22	33	38	22	22	25	22	145	74	85	62	82	22	33	34	38	22	26	91	118	39	25	91	25	38
16	33	85	38	22	25	22	38	26	74	25	62	25	22	91	22	26	91	33	51	25	34	94	49	91	26	25	22	87	22	22	91	22
17	62	25	65	85	26	49	33	39	22	65	25	22	34	34	65	26	34	25	25	26	22	82	123	22	25	106	46	85	39	133	33	133
18	22	33	25	22	26	38	25	25	25	57	85	26	22	25	22	22	26	22	91	39	38	46	38	85	133	33	51	49	85	22	65	22
19	49	62	38	34	39	87	22	91	22	33	38	26	49	26	25	133	25	146	91	106	22	25	25	25	22	46	34	25	57	25	22	26
20	22	22	65	26	22	62	39	58	85	49	91	39	25	26	85	22	39	22	38	51	46	33	58	26	38	22	26	33	25	58	25	34
21	94	142	33	25	25	25	22	49	65	22	49	25	57	39	26	91	22	74	26	25	38	49	22	39	22	25	25	85	65	86	22	278
22	38	22	57	39	22	106	51	33	49	26	25	91	65	33	26	65	91	25	25	33	49	22	26	49	25	22	39	22	85	38	91	91
23	25	26	22	46	65	33	25	25	25	22	26	85	85	25	39	25	22	121	22	69	91	26	22	91	57	26	33	91	33	26	91	25
24	51	39	26	38	34	49	65	22	133	82	22	33	46	22	25	49	25	26	49	22	25	22	25	25	34	26	91	22	65	25	133	46
25	34	58	22	33	145	22	58	46	26	91	39	38	25	51	33	34	51	26	22	65	26	62	206	65	91	39	22	38	22	33	25	33
26	49	74	39	25	25	25	301	22	26	25	22	22	26	22	49	91	34	34	57	22	26	91	69	22	85	25	25	26	86	51	26	85
27	26	25	145	22	57	22	46	121	34	49	25	65	22	46	22	33	57	25	25	85	39	46	91	341	25	49	22	26	22	22	39	22
28	25	38	25	34	62	26	25	25	22	46	49	22	39	25	133	25	46	85	133	74	22	49	65	22	26	34	133	34	26	91	33	25
29	22	33	91	22	33	39	62	26	49	87	65	51	22	86	22	22	25	22	46	26	25	25	25	25	26	33	65	25	26	22	38	22
30	82	218	65	49	26	529	22	34	22	33	62	34	25	91	38	26	33	65	65	26	22	85	33	91	22	85	91	91	25	106	22	85
31	22	22	226	25	22	25	26	87	65	25	46	25	91	62	91	22	26	22	49	25	65	33	65	38	86	22	25	33	169	86	65	26
32	91	25	33	74	39	34	22	57	58	22	25	26	65	26	58	217	22	25	25	38	321	34	22	26	22	57	206	49	38	46	22	26
33	25	22	25	26	22	65	25	25	25	185	33	39	49	25	49	25	39	65	51	33	34	22	91	26	34	22	26	22	74	85	49	25
34	62	57	22	26	91	33	111	46	65	22	91	205	34	34	25	91	22	51	22	91	25	25	22	25	49	49	26	25	33	25	62	38
35	85	26	38	39	91	133	38	22	34	26	22	33	25	22	26	65	86	34	26	22	85	22	26	33	69	202	39	22	25	34	25	91
36	65	26	22	25	25	22	46	86	49	25	26	25	33	85	33	38	85	69	22	25	49	26	26	49	38	25	22	34	22	26	34	33
37	46	25	26	46	86	38	65	22	33	46	22	22	38	22	85	58	65	25	25	22	33	39	38	22	25	26	91	86	34	26	169	34
38	25	49	25	22	46	22	25	25	25	58	38	34	22	25	22	25	87	26	49	34	26	178	34	65	74	39	22	85	22	22	26	22
39	133	51	39	65	65	133	33	34	22	34	87	22	26	38	25	46	25	39	38	82	22	25	25	22	33	158	38	25	85	25	25	49
40	22	33	51	22	33	26	34	58	39	62	34	49	22	133	22	22	38	22	58	91	38	51	57	94	169	33	46	26	25	22	25	22

means that 25 is the primary pretender for the bases that are 4th roots of 0 or 1 (mod 25) that have not already been coped with, that the first such base is 443, and that 1 in every 240.62 bases has 25 for its primary pretender (in fact 16 in every 3465 bases).

Another example is 'else 169 if $b^{12} \equiv 0$ or 1 (mod 169)', i.e., if $b \equiv \pm 19^e \pmod{169}$, for $1 \leq e \leq 6$ where the cases $e = 6$ ($b \equiv \pm 1$), $e = 3$ ($b \equiv \pm 70$), and $e = 2$ or 4 ($b \equiv \pm 23$ or ± 22) have already been preempted by $q_b = 26$ or 39, by 65, and by 91 respectively.

The largest first base is 10009487, for $q = 453$, while the greatest rarity is that of $q = 519$.

Reference

The paper was prompted by the table of pseudoprimes to various bases given by Albert H. Beiler on p. 42 of his *Recreations in the Theory of Numbers*, Dover, New York, 1964.

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Table 3: The first base and rarity of the 132 primary pretenders.

q	roots kth(m)	first base	rarity one in	q	roots kth(m)	first base	rarity one in	q	roots kth(m)	first base	rarity one in
4	1st(4)	0	2	159	2nd(53)	94763	83341.92	361	18th(361)	58727	159201.47
6	1st(3)	3	3	166	1st(83)	247838	69173.80	362	1st(181)	1050887	800429.64
9	2nd(9)	26	18	169	12th(169)	1202	22203.93	365	4th(73)	8222	313017.17
10	1st(5)	11	22.5	177	2nd(59)	111863	105468.69	381	2nd(127)	923162	1150798.45
14	1st(7)	14	52.5	178	1st(89)	48683	83809.94	382	1st(191)	383	886300.41
15	2nd(5)	59	63	183	2nd(61)	186842	113673.26	386	1st(193)	470342	905058.10
21	2nd(7)	83	157.5	185	4th(37)	1523	33318.02	393	2nd(131)	480638	1222539.21
22	1st(11)	23	216.56	194	1st(97)	58298	100995.26	394	1st(197)	384347	940782.12
25	4th(25)	443	240.62	201	2nd(67)	86027	138204.04	398	1st(199)	278402	960080.22
26	1st(13)	338	391.01	202	1st(101)	45047	109051.62	411	2nd(137)	786242	1315845.99
33	2nd(11)	263	639.84	205	4th(41)	14423	41858.20	417	2nd(139)	303158	1345305.23
34	1st(17)	578	679.83	206	1st(103)	32342	119760.96	422	1st(211)	231467	1043600.74
38	1st(19)	38	861.12	213	2nd(71)	53462	163633.79	427	6th(61)	149558	139812.54
39	2nd(13)	662	1114.39	214	1st(107)	79502	128741.29	445	4th(89)	739022	462456.86
46	1st(23)	47	1281.55	217	6th(31)	40883	17165.50	446	1st(223)	592958	1227712.86
49	6th(49)	227	854.36	218	1st(109)	37823	155920.00	447	2nd(149)	141698	1633246.98
51	2nd(17)	3467	2135.92	219	2nd(73)	169067	206921.87	451	10th(41)	18302	50339.80
57	2nd(19)	1823	2593.62	226	1st(113)	39098	167015.51	453	2nd(151)	10009487	2143037.38
58	1st(29)	842	2350.46	237	2nd(79)	141962	231715.22	454	1st(227)	283523	1643477.99
62	1st(31)	4898	2698.68	249	2nd(83)	357563	246959.64	458	1st(229)	277778	1672695.37
65	4th(13)	983	930.58	254	1st(127)	232538	196024.21	466	1st(233)	860702	1716907.59
69	2nd(23)	4622	4885.55	259	6th(37)	878	25091.09	469	6th(67)	473987	237840.01
74	1st(37)	4847	4519.13	262	1st(131)	162047	234781.00	471	2nd(157)	2264567	2457680.13
82	1st(41)	2747	5293.84	265	4th(53)	98663	91000.38	478	1st(239)	6085658	1907095.94
85	4th(17)	4127	1900.35	267	2nd(89)	232823	329876.41	481	12th(37)	108803	112655.45
86	1st(43)	11567	6809.60	274	1st(137)	112478	262750.39	482	1st(241)	2252387	2262497.08
87	2nd(29)	347	8968.74	278	1st(139)	27662	270535.59	485	4th(97)	968567	889852.41
91	6th(13)	542	689.90	289	16th(289)	197138	67140.22	489	2nd(163)	3166763	3114483.44
93	2nd(31)	17483	20007.20	291	2nd(97)	124547	398645.06	501	2nd(167)	4881242	3211811.04
94	1st(47)	2867	16791.76	298	1st(149)	142742	315947.41	502	1st(251)	1738427	2457818.82
106	1st(53)	22367	19776.96	301	6th(43)	32987	42986.04	505	4th(101)	2128262	967334.31
111	2nd(37)	43067	27144.85	302	1st(151)	150698	360605.13	511	6th(73)	210962	342597.56
118	1st(59)	18527	23552.15	303	2nd(101)	485102	479193.40	514	1st(257)	2338187	2751486.73
121	10th(121)	5042	9090.30	305	4th(61)	287138	141802.12	519	2nd(173)	1150103	3690229.26
122	1st(61)	5063	27725.42	309	2nd(103)	79103	511500.54	526	1st(263)	256163	2854500.87
123	2nd(41)	12422	36653.95	314	1st(157)	115238	401527.92	529	22nd(529)	37958	503092.10
129	2nd(43)	66047	39547.68	321	2nd(107)	41087	544005.57	537	2nd(179)	7345622	4047604.68
133	6th(19)	2858	3954.76	326	1st(163)	296987	426312.06	538	1st(269)	2735462	3093197.89
134	1st(67)	87302	44161.58	327	2nd(109)	10463	566650.81	542	1st(271)	183467	3139537.94
141	2nd(47)	11702	61146.80	334	1st(167)	127922	446371.15	543	2nd(181)	4503098	4178269.82
142	1st(71)	11147	49334.35	339	2nd(113)	851567	600572.10	545	4th(109)	4453598	1244091.57
145	4th(29)	3062	18589.75	341	10th(31)	2	16379.23	553	6th(79)	281738	454571.92
146	1st(73)	24602	56543.84	346	1st(173)	846662	708402.09	554	1st(277)	581423	3497678.40
158	1st(79)	158	62914.98	358	1st(179)	257402	741543.71	561	1st(1)	10103	25437.66