Kepler Confirmed

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One of the oldest unsolved problems in science may have finally been settled. On August 9, 1998, Thomas C. Hales announced [1] that he had proved Kepler's 1611 assertion that no packing of balls can be denser than the face-centered cubic lattice arrangement.

In this packing, seen in the piles of oranges in every grocer's shop, the spheres occupy .7405 of the space. Rogers remarked in 1958 [2] that "many mathematicians believe and every physicist knows" that no denser packing is possible. So why has it taken 387 years for a proof to be found?

Figure 1: Face-centered cubic lattice packing of cannon balls in Arlington, Virginia, c. 1863. [Mathew B. Brady or staff, Library of Congress.]

There are several reasons: there are technical difficulties caused by the fact that the density of a packing is defined as the *limit* of the fraction of space occupied by the balls as the number of balls goes to infinity. This means that (say) a million balls can be thrown away without changing the density!

Also, even if one considers only packings without any obvious gaps, there are still infinitely many different packings that are just as dense as the face-centered cubic lattice. These are the "Barlow packings", described in this journal in 1883 [3]: put down a layer of spheres arranged in the triangular lattice — the arrangement used when racking billiard balls — place another layer on top, and repeat. There are two ways to place each layer after the first, giving uncountably infinitely many distinct packings.

Thirdly, in the face-centered cubic lattice arrangement, each ball touches 12 others. However, there are infinitely many distinct ways to arrange 12 balls around another ball of the same size — there's a lot of slack, and there's almost room to squeeze in a thirteenth ball.

It is these various kinds of non-uniqueness that make the problem hard. (Paradoxically, the sphere-packing problems in four and eight dimensions may turn out to be easier, because the last two types of non-uniqueness are absent.) So far no one has found an elegant approach towards the three-dimensional problem. The methods that have been tried, and which in Hales's hands finally succeeded, are messy. The basic approach is to reduce a minimization problem involving infinitely many variables to a finite number of subproblems (and until the proof is finished it is never certain that the splitting into subproblems will terminate).

There have been two recent unsuccessful attacks on the problem that have received widespread attention. Buckminster Fuller claimed to give a proof in 1975 [4], but his arguments amount to a description of the face-centered cubic packing rather than a proof of its optimality.

In 1993 Wu-Yi Hsiang published a 100-page paper [5] claiming to give a proof. However, there are serious flaws in his arguments. To quote one review [6]: "If I am asked whether the paper fulfills what it promises in its title, namely a proof of Kepler's conjecture, my answer is: no. I hope that Hsiang will fill in the details, but I feel that the greater part of the work has yet to be done."

Hales has been working on the Kepler conjecture for ten years, and had proposed a five-step attack on the problem. Two parts have already been refereed and published [7]. The completed proof relies heavily on computers, which are used in several different ways. For example, at the heart of the proof are some 100000 linear programming problems, each involving 100 to 200 variables and 1000 to 2000 constraints. Hales has been careful to make all the computer calculations available on his web site [8]. Although the final acceptance of his claim must await a careful refereeing of the proof, there is little reason to doubt it. It is a considerable achievement.

There was never any real reason to suspect that the Kepler conjecture was false, although it was a little worrisome that there are packings of identical *ellipsoids* [9] that have density exceeding .75 .

Far from being the last word, Hales's result is just the beginning. Communication theorists as well as mathematicians are interested in determining the densest sphere packings in dimensions above 3. The sampling theorem of information theory [10] says that a signal containing no frequencies above W hertz can be reconstructed from its samples taken every $1/(2W)$ seconds. A signal that lasts for T seconds can therefore be represented by $2WT$ samples. Just as three numbers specify the coordinates of a point in 3-dimensional space, so these $2WT$ samples specify a point in $2WT$-dimensional space. The whole waveform is specified by a single point in $2WT$-dimensional space. Similar signals are represented by nearby points, dissimilar signals by well-separated points. One of the fundamental questions in communication theory is therefore to determine the densest packing of equal balls in n-dimensional space (where $n = 2WT$).

Incidentally, this geometrical way of representing signals, which is at the heart of Shannon's mathematical theory of communication [10], underlies the high-speed modems that we now take for granted. One of the most common coding schemes in use today works so well because the signals are represented as points in 8-dimensional space.

Many beautiful packings are known in higher dimensions, and have fascinating and unexpected connections with other branches of mathematics [11]. J. H. Conway and I recently described [12] what we think may be *all* the best sphere packings in dimensions up through 10, where "best" means both having the highest density and not containing any gaps. Let me conclude by mentioning an amazing property of what we conjecture are the densest 9-dimensional packings: in some of these packings (again there are infinitely many that are equally dense) half the spheres can be moved bodily through arbitrarily large distances without overlapping the other half, only touching them at isolated instants, and yet the density of the packing remains the same at all times! However, until someone extends Hales's result to higher dimensions, we have no proof that any packing in dimension greater than three is optimal.

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