A Linear Programming Bound for Orthogonal Arrays with Mixed Levels*

N. J. A. Sloane
Mathematical Sciences Research Center
AT&T Bell Laboratories
Murray Hill, NJ 07974

J. Stufken
Department of Statistics
Iowa State University
Ames, IA 50011

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Abstract

We show how the Delsarte theory can be used to obtain a linear programming bound for orthogonal arrays with mixed levels. Even for strength 2 this improves on the Rao bound in a large number of cases. The results point to several interesting sets of parameters for which the existence of the arrays is at present undecided.

Keywords: mixed orthogonal arrays; asymmetric orthogonal arrays; linear programming bound

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1. Introduction

Orthogonal arrays are extensively used in statistical experiments that call for fractional factorial designs. In such applications the columns correspond to the factors or variables in the experiment, and the rows specify the settings or level combinations of the factors at which observations are to be made. In most of the classical work on this subject it is assumed that all the factors have the same number of levels, that is, are "pure" or "unmixed" — see (Raghavarao, 1971), (Hedayat, Sloane and Stufken, 1995). For many applications, however, it is desirable to have orthogonal arrays in which different factors take on different numbers of levels. Examples of such "mixed-level" or "asymmetrical" orthogonal arrays (the precise definition is given in Section 2) can be found in (Addelman and Kempthorne, 1961), and they were formally defined in (Rao, 1973). In recent years several authors have discussed these arrays from various points of view: (Chacko et al., 1979), (Cheng, 1980, 1989), (Dey, 1985), (Gupta and Nigam, 1987), (Wu, 1989), (Wang and Safadi, 1990), (Safadi and Wang, 1991), (Wang and Wu, 1991, 1992), (Hedayat, Pu and Stufken, 1992), (Wu, Zhang and Wang, 1992), (Wang, 1995), and (Mukerjee and Wu, 1995).

The general theory developed by Delsarte (1973) provides bounds for codes and designs that are subsets of association schemes. In particular, it enables one to express the problem of finding the minimal number of runs in a "pure" orthogonal array of given strength as a linear programming problem.

In (Hedayat, Sloane and Stufken, 1995) we have evaluated this bound numerically, for a wide range of parameter values. The results are always at least as good as those obtained from the Rao (1947) bound (this is guaranteed by §5.3.2 of (Delsarte, 1973)), and are often much better. At first glance, however, Delsarte's theory does not seem to apply to mixed-level orthogonal arrays, since there is no scalar-valued definition of distance that makes the Cartesian product of a collection of Galois fields of different orders into an association scheme in any nontrivial way.

The way around this difficulty turns out to be simple: use a vector-valued definition of distance! This leads to versions of two of the main results of the Delsarte theory (see Theorems 1 and 2 below), and hence to a linear programming bound for mixed-level orthogonal arrays (Theorem 3). These results could be deduced from the general Delsarte theory, by determining the eigenvalues of the appropriate Cartesian product of association schemes. However, we prefer to give a direct proof, following the lines of the proof of the linear programming bound for codes given in MacWilliams and Sloane (1977, Chap. 5). These proofs are given in Section 3. The final section of the paper gives some comparisons of the new bound with the Rao bound. Even for strength 2 there are a large number of cases where the new bound is better.

2. The New Bound

Assume $r \geq 1$, $s_1, \ldots, s_r \geq 2$, $k_1, \ldots, k_r \geq 0$ and $t \geq 0$. A *mixed-level orthogonal array* $OA(N, s_1^{k_1} s_2^{k_2} \cdots s_r^{k_r}, t)$ *of strength* t is a matrix of size $N \times k$, where $k = k_1 + \cdots + k_r$, in which k_1 of the columns have entries from $\{0, 1, \ldots, s_1 - 1\}$, k_2 of the columns have entries from $\{0, 1, \ldots, s_2 - 1\}$, etc., with the property that in any $N \times t$ subarray all t-tuples that could occur as rows appear equally often. In the present paper we will assume that $s_1, \ldots, s_r$ are distinct. If $r = 1$ the array is called *pure* or *unmixed* rather than mixed.

To avoid unnecessary complications we shall only state and prove our theorems for the case $r = 2$. The general case is precisely analogous.

We will therefore consider matrices M of size $N \times k$, where $k = k_1 + k_2$, and furthermore we will assume that the symbols in the first k_1 columns are from $\{0, \ldots, s_1 - 1\}$ and that those in the last k_2 columns are from $\{0, \ldots, s_2 - 1\}$. We will refer to an arbitrary matrix M with these properties as an $OA(N, s_1^{k_1} s_2^{k_2}, 0)$, i.e. strength 0 means that the columns are otherwise unrestricted.

The rows of M are therefore elements of $\mathbb{Z}_{s_1}^{k_1} \times \mathbb{Z}_{s_2}^{k_2}$, where $\mathbb{Z}_s$ denotes a cyclic group of order s. A typical row will be written as

$$u = u_1 u_2 \ , \tag{1}$$

where

$$u_1 = u_{11}u_{12}\cdots u_{1k_1} \in \mathbb{Z}_{s_1}^{k_1} , \tag{2}$$

$$u_2 = u_{21}u_{22}\cdots u_{2k_2} \in \mathbb{Z}_{s_2}^{k_2} . \tag{3}$$

The *Hamming weight*, $wt(u)$, of a vector u is the number of nonzero components, and the *Hamming distance*, $\mathrm{dist}(u,v)$, between two vectors of the same length is the number of places where they differ.

If $u = u_1u_2$ is a row of M, we let $A_{i_1,i_2}(u)$ denote the number of rows $v = v_1v_2$ of M with $\mathrm{dist}(u_1,v_1) = i_1$ and $\mathrm{dist}(u_2,v_2) = i_2$. The *distance distribution* of M is the $(k_1+1)\times(k_2+1)$ matrix $A = \{A_{i_1,i_2} : 0 \le i_1 \le k_1,\ 0 \le i_2 \le k_2\}$, where

$$A_{i_1,i_2} = \frac{1}{N}\sum_u A_{i_1,i_2}(u) ,$$

the sum being taken over all rows of M. The *distance enumerator* for M is then the polynomial

$$W(x_1,y_1,x_2,y_2) = \sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2} A_{i_1,i_2} x_1^{k_1-i_1} y_1^{i_1} x_2^{k_2-i_2} y_2^{i_2} . \tag{4}$$

Note that $A_{0,0} \ge 1$, with equality if and only if the rows of the array are distinct.

We also define the *MacWilliams transform* of the distance distribution to be the $(k_1+1)\times(k_2+1)$ matrix

$$A' = \{A'_{j_1,j_2} : 0 \le j_1 \le k_1, 0 \le j_2 \le k_2\} ,$$

where

$$\frac{1}{N}W(x_1+(s_1-1)y_1, x_1-y_1, x_2+(s_2-1)y_2, x_2-y_2)$$
$$= \sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k_2} A'_{j_1,j_2} x_1^{k_1-j_1} y_1^{j_1} x_2^{k_2-j_2} y_2^{j_2} . \tag{5}$$

The relationship between the numbers $\{A_{i_1,i_2}\}$ and $\{A'_{j_1,j_2}\}$ may be expressed in terms of Krawtchouk polynomials. For integers $s \ge 2$ and $m \ge 0$ we define the *Krawtchouk polynomial* $P_j^{(s)}(x;m)$ to be

$$\sum_{\nu=0}^{j}(-1)^\nu (s-1)^{j-\nu}\binom{x}{\nu}\binom{m-x}{j-\nu} ,$$

for $j = 0,1,\ldots,m$ (Szegö, 1959), (Abramowitz and Stegun, 1964), (MacWilliams and Sloane, 1977). Then

$$A'_{j_1,j_2} = \frac{1}{N}\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2} P_{j_1}^{(s_1)}(i_1;k_1)P_{j_2}^{(s_2)}(i_2;k_2)A_{i_1,i_2} , \tag{6}$$

for $0 \le j_1 \le k_1$, $0 \le j_2 \le k_2$. That (6) is equivalent to (4) and (5) follows immediately from the generating functions for Krawtchouk polynomials (cf. MacWilliams and Sloane, 1977, Chap. 5, Theorem 3).

Finally, we define the *dual distance* $d^\perp$ of M as follows. If some A'_{j_1,j_2} is positive, for $0 \le j_1 \le k_1$, $0 \le j_2 \le k_2$, $0 < j_1 + j_2$, we set

$$d^\perp = \min\{j_1 + j_2 : 0 \le j_1 \le k_1, 0 \le j_2 \le k_2, j_1 + j_2 > 0, A'_{j_1,j_2} > 0\} ;$$

otherwise we set $d^\perp = k_1 + k_2 + 1$.

We can now state the two main results of this paper.

Theorem 1. *Let M be an $OA(N, s_1^{k_1} s_2^{k_2}, 0)$, with distance distribution $\{A_{i_1,i_2}\}$. Then the MacWilliams transform of the distance distribution satisfies*

$$A'_{j_1,j_2} \ge 0, \quad 0 \le j_1 \le k_1, \quad 0 \le j_2 \le k_2 \ .$$

Theorem 2. *Let M be as in Theorem 1. Then M is an $OA(N, s_1^{k_1} s_2^{k_2}, t)$ if and only if $d^\perp \ge t + 1$.*

To illustrate these theorems, consider the $OA(18, 2^1 3^7, 2)$ given for example in (Taguchi, 1987, p. 1153). The numbers $A_{i_1,i_2}(u)$ are independent of the choice of u, the distance distribution is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 & 6 & 0 & 0 \end{bmatrix} ,$$

and the MacWilliams transform is

$$A' = \begin{bmatrix} 1 & 0 & 0 & 22 & 34\,{}^1\!/_2 & 27 & 31 & 6 \\ 0 & 0 & 6 & 18 & 25\,{}^1\!/_2 & 39 & 27 & 6 \end{bmatrix} ,$$

where the indices run from 0 to 1 and 0 to 7 in both cases. Then indeed all A'_{j_1,j_2} are nonnegative, and $A'_{0,1} = A'_{1,0} = A'_{0,2} = A'_{1,1} = 0$, so that $d^\perp = 3$, in accordance with Theorems 1 and 2.

The linear programming bound is an immediate consequence of Theorems 1 and 2.

Theorem 3. *Given s_1, k_1, s_2, k_2, t with $s_1, s_2 \ge 2$, $s_1 \ne s_2$, $k_1, k_2 \ge 0$, $t \ge 1$, choose real numbers A_{i_1,i_2} $(0 \le i_1 \le k_1, 0 \le i_2 \le k_2)$ so as to minimize*

$$\sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} A_{i_1,i_2} \tag{7}$$

subject to

$$
\begin{aligned}
A_{0,0} &\geq 1, \\
A_{i_1,i_2} &\geq 0, \quad \textit{for} \quad 0 \leq i_1 \leq k_1, \quad 0 \leq i_2 \leq k_2, \\
A'_{j_1,j_2} &\geq 0, \quad \textit{for} \quad 0 \leq j_1 \leq k_1, \quad 0 \leq j_2 \leq k_2, \\
A'_{j_1,j_2} &= 0, \quad \textit{for} \quad 1 \leq j_1 + j_2 \leq t,
\end{aligned}
$$

where the A'_{j_1,j_2} *are given by (6). Let* N_{LP} *be the minimal value of (7). Then any* $OA(N, s_1^{k_1} s_2^{k_2}, t)$ *satisfies* $N \geq N_{LP}$.

The number of rows in an $OA(N, s_1^{k_1} s_2^{k_2}, t)$ must be divisible by

$$
N_0 = l.c.m. \left\{ s_1^{i_1} s_2^{i_2} : 0 \leq i_1 \leq k_1, 0 \leq i_2 \leq k_2, 0 \leq i_1 + i_2 \leq t \right\} \ . \tag{8}
$$

So if N_{LP} is not divisible by N_0, we can increase the lower bound to the next smallest multiple of N_0.

3. Proofs of Theorems 1 and 2

The proofs are parallel to those of Theorems 6 and 7 of (MacWilliams and Sloane, 1977, Chap. 5), and we shall be brief. We begin by introducing some terminology, the purpose of which is to reduce the hard work to a series of verifications.

Let R denote the polynomial ring $\mathbb{C}[z_1, \ldots, z_{k_1}, Z_1, \ldots, Z_{k_2}]/I$, where I is the ideal generated by

$$
z_i^{s_1} - 1, \ 1 \leq i \leq k_1; \ Z_j^{s_2} - 1, \ 1 \leq j \leq k_2 \ .
$$

We define a conjugation operation on R, denoted by a bar, which replaces each coefficient by its complex conjugate and replaces each z_i by $z_i^{s_1-1}$, and each Z_j by $Z_j^{s_2-1}$.

To each element $u = u_1 u_2 \in \mathbb{Z}_{s_1}^{k_1} \times \mathbb{Z}_{s_2}^{k_2}$, given by (1)–(3), we associate the element

$$
\phi(u) = \prod_{i=1}^{k_1} z_i^{u_{1i}} \cdot \prod_{j=1}^{k_2} Z_j^{u_{2j}}
$$

of R, which we abbreviate to

$$
z^{u_1} Z^{u_2} \ .
$$

A typical element of R has the form

$$
C = \sum_{u_1 \in \mathbb{Z}_{s_1}^{k_1}} \sum_{u_2 \in \mathbb{Z}_{s_2}^{k_2}} c_{u_1 u_2} z^{u_1} Z^{u_2} \ , \tag{9}
$$

where $c_{u_1 u_2} \in \mathbb{C}$, and we set $|C| = \sum\sum c_{u_1 u_2}$.

To each $v = v_1 v_2 \in \mathbb{Z}_{s_1}^{k_1} \times \mathbb{Z}_{s_2}^{k_2}$ we associate the map $\chi_{v_1 v_2} : R \to \mathbb{C}$ defined on monomials by

$$\chi_{v_1 v_2}(z^{u_1} Z^{u_2}) = \xi_1^{v_1 \cdot u_1} \xi_2^{v_2 \cdot u_2} \ ,$$

where $\xi_1 = e^{2\pi i/s_1}$, $\xi_2 = e^{2\pi i/s_2}$, and the exponents are the standard dot products in $\mathbb{Z}_{s_1}^{k_1}$ and $\mathbb{Z}_{s_2}^{k_2}$, respectively. We extend the map to arbitrary elements (9) of R by linearity.

The Fourier transform of $C \in R$ with $|C| \neq 0$ is defined to be

$$C' = \frac{1}{|C|} \sum_{v_1 \in \mathbb{Z}_{s_1}^{k_1}} \sum_{v_2 \in \mathbb{Z}_{s_2}^{k_2}} \chi_{v_1 v_2}(C) z^{v_1} Z^{v_2} \ . \tag{10}$$

Finally, we associate with our matrix M the element

$$C = \sum_u z^{u_1} Z^{u_2} \in R \ ,$$

where the sum is over all rows $u = u_1 u_2$ of M, and set

$$D = \frac{1}{N} C \overline{C} \ ,$$

where the bar is the conjugation operator defined above.

Now that this terminology has been established, we can give the proof of Theorem 1. The following verifications are straightforward and we omit the details.

(i) If we write

$$D = \sum_{u_1 \in \mathbb{Z}_{s_1}^{k_1}} \sum_{u_2 \in \mathbb{Z}_{s_2}^{k_2}} d_{u_1 u_2} z^{u_1} Z^{u_2}$$

then

$$A_{i_1, i_2} = \sum_{wt(u_1) = i_1} \sum_{wt(u_2) = i_2} d_{u_1 u_2} \ .$$

(ii) If we write the Fourier transform

$$D' = \sum_{v_1 \in \mathbb{Z}_{s_1}^{k_1}} \sum_{v_2 \in \mathbb{Z}_{s_2}^{k_2}} d'_{v_1 v_2} z^{v_1} Z^{v_2}$$

then

$$A'_{j_1, j_2} = \sum_{wt(v_1) = j_1} \sum_{wt(v_2) = j_2} d'_{v_1 v_2} \ .$$

(iii) Hence

$$\begin{aligned} A'_{j_1,j_2} &= \frac{1}{N^2} \sum_{wt(v_1)=j_1} \sum_{wt(v_2)=j_2} \chi_{v_1 v_2}(C\overline{C}) \\ &= \frac{1}{N^2} \sum_{wt(v_1)=j_1} \sum_{wt(v_2)=j_2} |\chi_{v_1 v_2}(C)|^2 \geq 0 \ , \end{aligned}$$

which establishes Theorem 1.

For Theorem 2 we need the following elementary result. This is a straightforward generalization of a standard fact about pure orthogonal arrays, see Theorem 3.26 of (Hedayat, Sloane and Stufken, 1995), and we omit the proof.

Lemma 1. *Let M be as in Theorems 1 and 2, and let $\xi_1 = e^{2\pi i/s_1}$ and $\xi_2 = e^{2\pi i/s_2}$. Then M is an $OA(N, s_1^{k_1} s_2^{k_2}, t)$ if and only if*

$$\sum_{u = \text{row of } M} \xi_1^{v_1 \cdot u_1} \xi_2^{v_2 \cdot u_2} = 0 \tag{11}$$

for all $v = v_1 v_2 \in \mathbb{Z}_{s_1}^{k_1} \times \mathbb{Z}_{s_2}^{k_2}$ with

$$1 \leq wt(v_1) + wt(v_2) \leq t \ .$$

Suppose now that the dual distance $d^{\perp}$ of M is at least $t+1$. Therefore $A'_{j_1,j_2} = 0$ for all j_1, j_2 with $1 \leq j_1 + j_2 \leq t$. From (iii), $\chi_{v_1 v_2}(C) = 0$ for all v_1, v_2 with $1 \leq wt(v_1) + wt(v_2) \leq t$. By the Lemma, M has strength t. The proof of the reverse implication is similar.

4. Numerical Examples and Comparison with Rao Bound

We discuss four families of orthogonal arrays, $OA(N, 2^{k_1} 3^{k_2}, t)$, for $t = 2, 3, 4$, and $OA(N, 2^{k_1} 4^{k_2}, 2)$, and compare the results obtained from the linear programming bound of Theorem 3 with the appropriate generalization of the Rao bound. We assume that both k_1 and k_2 are positive, so that the array is indeed mixed. For a comparison of the bounds in the case of "pure" orthogonal arrays see Hedayat, Sloane and Stufken (1995).

Let $\gamma_1 = s_1 - 1$, $\gamma_2 = s_2 - 1$, and suppose $s_1 < s_2$. Then the generalized Rao bound for an $OA(N, s_1^{k_1} s_2^{k_2}, t)$ with $t = 2, 3, 4$ states that

$$N \geq 1 + k_1\gamma_1 + k_2\gamma_2, \quad \text{for } \ t = 2, \tag{12}$$

$$N \geq 1 + k_1\gamma_1 + k_2\gamma_2 + k_1\gamma_1\gamma_2 + (k_2 - 1)\gamma_2^2, \quad \text{for } \ t = 3 \ , \tag{13}$$

and

$$N \geq 1 + k_1\gamma_1 + k_2\gamma_2 + \binom{k_1}{2}\gamma_1^2 + k_1k_2\gamma_1\gamma_2 + \binom{k_2}{2}\gamma_2^2, \quad \text{for} \quad t = 4 \tag{14}$$

(see Hedayat, Sloane and Stufken, 1995). We let N_{Rao} denote the bound given by (12)–(14), and define N^*_{LP} (resp. N^*_{Rao}) to be the value obtained by rounding N_{LP} (resp. N_{Rao}) up to the next multiple of N_0 (see (8)).

The values of the linear programming bound were obtained using the CPLEX optimizer (CPLEX, 1991), via a program written in the AMPL mathematical programming language (Fourer, Gay and Kernighan, 1993).

$\mathbf{OA(N, 2^{k_1}3^{k_2}, 2)}$. The values of N_{LP} were computed for arrays of type $OA(N, 2^{k_1}3^{k_2}, 2)$ for the range $k_1 + 2k_2 \leq 70$, $k_1 \leq 60$. (Outside this range the coefficients in the linear program get too large.)

In this range N^*_{LP} and N^*_{Rao} generally agree, except that $N^*_{LP} > N^*_{Rao}$ in the following cases:

(i) $k_1 = 1$, $k_2 = 9m - 1$, $m \geq 1$, when $N^*_{LP} = N^*_{Rao} + 18$;

(ii) $k_2 = 1$, $k_1 = 12m - 3$, $m \geq 1$, when $N^*_{LP} = N^*_{Rao} + 12$;

(iii) $k_1 = 2m + 1$, $k_2 = 17 - m$, $m = 1, 2, \ldots, 15$, $m \neq 5$, when $N^*_{LP} = 72$ but $N^*_{Rao} = 36$;

(iv) $k_1 = 2m + 1$, $k_2 = 35 - m$, $m = 1, 2, \ldots, 29$, $m \neq 5$, 11, 12, 14, 17, 18, 20, 23, when $N^*_{LP} = 108$ but $N^*_{Rao} = 72$;

(v) $k_1 = 30$, $k_2 = 2$, when $N^*_{LP} = 72$ and $N^*_{Rao} = 36$.

The values of m in (iii) and (iv) where N^*_{LP} and N^*_{Rao} agree are especially interesting. These are cases when $N^*_{LP} = N_{LP} = N_{Rao} = N^*_{Rao}$, and point to sets of parameters where unusually good orthogonal arrays *might* exist. In some cases these arrays are already known, for example arrays $OA(36, 2^{11}3^{12}, 2)$, $OA(72, 2^{23}3^{24}, 2)$ and $OA(72, 2^{47}3^{12}, 2)$ can be obtained from Taguchi (1987), Dey (1985), and Wang (1995). An alternative way to obtain these three arrays is the following. Let D_1 denote a difference scheme $D(24, 24, 3)$ and let E_1 denote an $OA(24, 2^{23}, 2)$. Let D_2 denote a difference scheme $D(36, 36, 2)$ and let E_2 denote an $OA(36, 2^{11}3^{12}, 2)$. Then the matrices

$$\begin{bmatrix} D_1 & E_1 \\ D_1 + 1 & E_1 \\ D_1 + 2 & E_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} D_2 & E_2 \\ D_2 + 1 & E_2 \end{bmatrix}$$

are respectively orthogonal arrays $OA(72, 2^{23}3^{24}, 2)$ and $OA(72, 2^{47}3^{12}, 2)$ (cf. Hedayat, Pu and Stufken, 1992; Wang and Wu, 1991). An $OA(36, 2^{11}3^{12}, 2)$ can be similarly obtained from a difference scheme $D(12, 12, 3)$ and an $OA(12, 2^{11}, 2)$. An $OA(72, 2^{47}3^{12}, 2)$ can also be found in Wang (1995). In the remaining cases, however, $OA(72, 2^{11}3^{30}, 2)$, $OA(72, 2^{25}3^{23}, 2)$, $OA(72, 2^{29}3^{21}, 2)$, $OA(72, 2^{35}3^{18}, 2)$, etc., we do not know if an orthogonal array exists.

It is worth mentioning that the results in (iii) give a simple proof of a result of Mukerjee and Wu (1995). These authors showed with some difficulty that the maximal value of k_2 in an $OA(36, 2^{35-2k_2}3^{k_2}, 2)$ is 12. The results in (iii) show immediately that 12 is the *only* value of k_2 for which an $OA(36, 2^{35-2k_2}3^{k_2}, 2)$ can exist.

The linear programming bound can also provide information that may be useful in attempting to construct, or disprove the existence of, these arrays. For example, CPLEX reports the distance distribution $\{A_{0,0} = 1,\ A_{0,24} = 5, A_{6,20} = 66\}$, the other $A_{i,j}$ being zero, for a possible $OA(72, 2^{11}3^{30}, 2)$. It would be nice to know if such an array exists, since it would be a remarkable analogue of a "two-weight code", cf. Calderbank and Kantor (1986).

The following argument shows that if an $OA(72, 2^{11}3^{30}, 2)$ does exist then its distance distribution must be as stated. From Lemma 1 of Mukerjee and Wu (1995) we find that if $A_{i,j} > 0$ then either $(i, j) = (0, 0)$ or $2i + 3j = 72$, so the only possible nonzero $A_{i,j}$'s are $A_{0,0}$, $A_{0,24}$, $A_{3,22}$, $A_{6,20}$ and $A_{9,18}$. Furthermore these are rational numbers with denominator dividing 36. We find by computer that if $A_{3,22} \geq 1/36$ then the number of runs is at least 73. Hence $A_{3,22} = 0$. Similarly $A_{9,18} = 0$, $A_{0,24} = 5$ and $A_{6,20} = 66$.

$\mathbf{OA(N, 2^{k_1}4^{k_2}, 2)}$. We computed N_{LP} for $k_1 + 3k_2 \leq 65$, and for a number of other sets of parameters. The results indicate that again N^*_{LP} and N^*_{Rao} generally agree, except in the following cases where $N^*_{LP} = N^*_{Rao} + 16$:

(i) $k_1 = 1$, $k_2 = 16m + 10$ and $16m + 15$, $m \geq 0$;

(ii) $k_1 = 2$, $k_2 = 16m + 14$, $m \geq 0$.

$\mathbf{OA(N, 2^{k_1}3^{k_2}, 3)}$. The values of N_{LP} were computed for $k_1 + 2k_2 \leq 75$, $k_1 \leq 60$. N^*_{LP} and N^*_{Rao} agree except in the following cases:

(i) $k_1 = 1$, $k_2 = 9m$, $m \geq 1$, when $N^*_{LP} = N^*_{Rao} + 54$;

(ii) $k_2 = 2$, $k_1 = 24m + 21$, $m \geq 0$, when $N^*_{LP} = N^*_{Rao} + 72$;

(iii) $k_1 = 2m+1$, $k_2 = 36-m$, for $m = 1, 2, \ldots, 29$, $m \neq 12, 18$, when $N^*_{LP} = 432$ but $N^*_{Rao} = 216$.

The two exceptional cases in (iii), where $N_{LP} = N_{Rao} = 216$, would correspond to arrays $OA(216, 2^{25}3^{24}, 3)$ and $OA(216, 2^{37}3^{18}, 3)$. Do these exist?

$\mathbf{OA(N, 2^{k_1}3^{k_2}, 4)}$. For arrays of strength 4 there is a more pronounced difference between the two bounds, as is illustrated in Table 1, which shows the values of N^*_{LP}, N_{LP}, N^*_{Rao}, N_{Rao} for arrays $OA(N, 2^{k_1}3^1, 4)$ with $3 \leq k_1 \leq 20$. As can be seen, the new bound is often significantly better. Note that $N^*_{LP} = N_{LP} = N^*_{Rao} = 144$ when $k_1 = 13$. Does an $OA(144, 2^{13}3^1, 4)$ exist?

Table 1: Comparison of linear programming and Rao bounds for arrays $OA(N, 2^{k_1}3^1, 4)$.

k_1	$N^*_{LP}(N_{LP})$	$N^*_{Rao}(N_{Rao})$
3	24(24)	24(15)
4	48(24)	48(21)
5	48(36)	48(28)
6	48(53.33)	48(36)
7	96(74.67)	48(45)
8	96(85.33)	96(55)
9	144(100.2)	96(66)
10	144(105.4)	96(78)
11	144(121.0)	96(91)
12	144(140.4)	144(105)
13	144(144)	144(120)
14	192(157.8)	144(136)
15	192(178.9)	192(153)
16	240(198.3)	192(171)
17	240(223.2)	192(190)
18	240(234.3)	240(210)
19	288(264.3)	240(231)
20	336(302.0)	288(253)

Postscript

Exact or analytical — as opposed to computational — versions of the linear programming bound have recently been obtained by J. Millar and N. J. A. Sloane. The results confirm and extend those in Section 4. Details will appear elsewhere.

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