

References

- [1] E. F. Assmus, Jr., and H. F. Mattson, Jr., personal communication.
- [2] J. H. Conway and N. J. A. Sloane, *Sphere-Packings, Lattices and Groups*, Springer-Verlag, NY, 1988.
- [3] A. G. Earnest and G. Nipp, “On the theta series of positive quaternary quadratic forms,” *C. R. Math. Rep. Acad. Sci. Canada*, **13** (No. 1, Feb. 1991), 33–38.
- [4] C. S. Gordon, D. L. Webb and S. A. Wolpert, “You can’t hear the shape of a drum,” preprint.
- [5] M. Kac, “Can one hear the shape of a drum?,” *Amer. Math. Monthly*, **73** (1966), 1–23.
- [6] M. Kneser, “Lineare Relationen zwischen Darstellungsanzahlen quadratischen Formen,” *Math. Ann.*, **168** (1967), 31–39.
- [7] Y. Kitaoka, “Positive definite forms with the same representation numbers,” *Arch. Math.*, **28** (1977), 495–497.
- [8] J. Milnor, “Eigenvalues of the Laplace operator on certain manifolds,” *Proc. Nat. Acad. Sci. U.S.A.*, **51** (1964), 542.
- [9] A. Schiemann, “Ein Beispiel positiv definiter quadratischer Formen der Dimension 4 mit gleichen Darstellungszahlen,” *Arch. Math.*, **54** (1990), 372–375.
- [10] R. Schulze-Pillot, personal communication, 1991.
- [11] N. J. A. Sloane, “Self-dual codes and lattices,” in *Relations between Combinatorics and Other Parts of Mathematics*, Proc. Symp. Pure Math., Vol. **34**, 1979, Amer. Math. Soc., Providence RI, pp. 273–308.
- [12] G. L. Watson, “Determination of a binary quadratic form by its values at integer points,” *Mathematika*, **26** (1979), 72–75 and **27** (1980), 188.
- [13] E. Witt, “Eine Identität zwischen Modulformen zweiten Grades,” *Abh. Math. Sem. Hamburg*, **14** (1941), 323–337.

obtain 3-dimensional lattices. Unfortunately these turn out to be equivalent lattices. Furthermore the sublattices of $L^\sigma(a, b, c, d)$ spanned by the even sums of $v_\infty^\sigma, v_0^\sigma, v_1^\sigma, v_2^\sigma$ are also equivalent.

(ix) The lattices $L^\sigma(a, b, c, d)$ were found from Schiemann's first example [9] by a computational process that will be described in a separate publication. They may be regarded as lattice versions of the tetracode (cf. [2, p. 81]).

(x) There also appear to be several 2-parameter families of isospectral 4-dimensional lattices, for instance the pair defined by the Gram matrices

$$\begin{bmatrix} a+6b & -5b & 5b & 2b \\ -5b & a+2b & 6b & 5b \\ 5b & 6b & a-2b & 5b \\ 2b & 5b & 5b & a-6b \end{bmatrix}, \quad \begin{bmatrix} a+6b & -2b & b & 7b \\ -2b & a+2b & 9b & b \\ b & 9b & a-2b & 2b \\ 7b & b & 2b & a-6b \end{bmatrix},$$

of determinant $(a^2 - 90b^2)^2$, where $a > \sqrt{90}|b|$. These can be made arbitrarily close to the cubic lattice. They are classically integral if a and b are integers. This family yields isospectral classically integral lattices of determinants 31^2 (which we suspect is the smallest possible), 79^2 , 81^2 , $106^2, \dots$. There are similar 2-parameter families whose determinants are perfect squares that include more cases from Schiemann's list.

3. Isospectral 5- and 6-dimensional lattices

The two binary self-dual codes of length 6 generated by 110000, 001100, 000011 and 110000, 101000, 111111 both have weight enumerator $(x^2 + y^2)^3$. (They were apparently first discovered by Assmus and Mattson in the 1960's [1]. Self-dual codes with weight enumerator $(x^2 + y^2)^{n/2}$ were classified for lengths $n \leq 16$ in [11].) By applying Construction A ([2],[11]) one obtains a pair of isospectral 6-dimensional lattices with theta series $\vartheta_3(q^2)^6 = 1 + 12q^2 + 64q^4 + \dots$. These are integral lattices of determinant 64, one of which is the rescaled cubic lattice $\sqrt{2}I_6$. Our method shows that this pair lies in a 9-parameter family of isospectral pairs.

A pair of isospectral (but inequivalent) 5-dimensional lattices are then obtained by taking the vectors perpendicular to the all-1's vector in these two lattices. They have Gram matrices

$$\begin{bmatrix} 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 8 & 4 \\ 2 & 0 & 2 & 4 & 8 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 6 & 4 & 4 \\ 2 & 2 & 4 & 8 & 4 \\ 2 & 2 & 4 & 4 & 8 \end{bmatrix}$$

and determinant 96.

correspondence between the vectors $[w, x, y, z]$ of L^+ and those of L^- by changing the sign of the first coordinate of $[w, x, y, z]$ that is divisible by 3.

Remarks.

(i) It is easy to see that $L^\pm(a, b, c, d)$ is a classically integral lattice just if a, b, c, d are integers congruent modulo 6 whose sum is divisible by 4, and an even or Type II lattice if $a + b + c + d$ is divisible by 8.

(ii) $L^\sigma(a, b, c, d) \cong L^\sigma(a, d, b, c) \cong L^{-\sigma}(b, a, c, d)$, where \cong indicates equivalent lattices. So if any two of a, b, c, d are equal then $L^+(a, b, c, d) \cong L^-(a, b, c, d)$.

(iii) The first nontrivial classically integral case is therefore $a = 1, b = 7, c = 13, d = 19$, which in fact gives Schiemann's initial pair of lattices [9] of determinant $1 \cdot 7 \cdot 13 \cdot 19 = 1729$. To show that these lattices are isospectral, Schiemann verified by computer that the first 750 terms of their theta series agreed, and then used the theory of modular forms. The above proof is simpler and more general.

Other small values of a, b, c, d (1,7,13,43; 1,7,25,31; etc.) give all those of Schiemann's first dozen examples whose determinant is not a perfect square. By taking $a, b, c, d = 7, 13, 19, 49$ we obtain Earnest and Nipp's pair [3].

(iv) By taking $a, b, c, d = 1, 7, 13, 31; 1, 7, 19, 25; 2, 8, 14, 20; \text{etc.}$ we obtain pairs of odd or Type I isospectral lattices of determinants 2821, 3325, 4480, 5005, 6517, etc. These appear to be new. (Schiemann's search only covered even lattices.)

(v) We conjecture that $L^+(a, b, c, d)$ and $L^-(a, b, c, d)$ are inequivalent whenever $a < b < c < d$. We have verified this for the classically integral lattices whose determinants $abcd$ are less than 10000.

(vi) The dual of $L^\sigma(a, b, c, d)$ is $L^{-\sigma}(a^{-1}, b^{-1}, c^{-1}, d^{-1})$.

(vii) By making a, b, c, d nearly equal we obtain pairs of isospectral lattices that are arbitrarily close to the cubic lattice I_4 . In the next section we see that in 6 dimensions the cubic lattice I_6 is itself part of an isospectral pair. On the other hand we have proved that there does not exist a pair of 3-dimensional isospectral lattices that are arbitrarily close to I_3 , nor a pair of 5-dimensional isospectral lattices one of which is I_5 .

(viii) If we set $a = 0$ in $L^\pm(a, b, c, d)$ and neglect vectors of zero length in the usual way, we

In 1986 one of the present authors observed that pairs of isospectral lattices in 6 and 5 dimensions could be obtained from a pair of codes with the same weight enumerator given by the other author [11]. These lattices, mentioned in [2, p 47] but hitherto unpublished, are given in Section 3.

The first pair of isospectral 4-dimensional lattices was found in 1990 by Schiemann [9], by computer search, and we have been informed by Schulze-Pillot [10] that Schiemann has since found at least a dozen such pairs. Another pair has been given by Earnest and Nipp [3].

On the other hand it is known that in 2 dimensions a lattice is determined by its theta series (see Watson [12] and the references given therein). In 3 dimensions the problem is open. The answer to the representation-number question stated in the second sentence is therefore either 2 or 3.

The main purpose of the present chapter is to give a simple 4-parameter family of isospectral 4-dimensional lattices. Seven of Schiemann's first dozen pairs, including the original pair, occur as special cases. Our construction provides some geometric understanding of these computer-generated examples.

2. A 4-parameter family of isospectral 4-dimensional lattices

Theorem 1. *Let e_∞, e_0, e_1, e_2 be orthogonal vectors satisfying*

$$e_\infty \cdot e_\infty = \frac{a}{12}, \quad e_0 \cdot e_0 = \frac{b}{12}, \quad e_1 \cdot e_1 = \frac{c}{12}, \quad e_2 \cdot e_2 = \frac{d}{12},$$

where $a, b, c, d > 0$, and let $[w, x, y, z]$ denote the vector $w e_\infty + x e_0 + y e_1 + z e_2$. Let

$$v_\infty^\pm = [\pm 3, -1, -1, -1], \quad v_0^\pm = [1, \pm 3, 1, -1], \quad v_1^\pm = [1, -1, \pm 3, 1],$$

$$v_2^\pm = [1, 1, -1, \pm 3].$$

Then the lattices $L^+(a, b, c, d)$ spanned by $v_\infty^+, v_0^+, v_1^+, v_2^+$ and $L^-(a, b, c, d)$ spanned by $v_\infty^-, v_0^-, v_1^-, v_2^-$ are isospectral. Both have determinant $abcd$.

Sketch of proof. For $\sigma = +$ or $-$, it is easy to see that $L^\sigma(a, b, c, d)$ has a sublattice $M^\sigma(a, b, c, d)$ spanned by those vectors $(\pm 3, \pm 3, \pm 3, \pm 3)$ in which the product of the signs is σ . Changing the sign of any coordinate of a vector of $M^\sigma(a, b, c, d)$ yields a vector of $M^{-\sigma}(a, b, c, d)$. Representatives for the nontrivial cosets of M^σ in L^σ are the four vectors $v_\infty^\sigma, v_0^\sigma, v_1^\sigma, v_2^\sigma$ and their negatives. It follows that we obtain a one-to-one length-preserving

Four-Dimensional Lattices With the Same Theta Series*

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ABSTRACT

This paper constructs a simple 4-parameter family of pairs of 4-dimensional lattices that have the same theta series, i.e. are isospectral. Many of the known examples of isospectral lattices occur as special cases, as well as new classically integral examples with determinants 2821, 3325, There are also a number of 2-parameter families whose determinants are perfect squares, the smallest of which is 961. Both the 4-parameter family and one of the 2-parameter families include isospectral pairs that are arbitrarily close to the cubic lattice I_4 . Isospectral pairs of lattices of determinants 96 and 64 are given in dimensions 5 and 6, respectively. One lattice of the 6-dimensional pair is a scaled version of the cubic lattice I_6 .

1. Introduction

The recent solution by Gordon, Webb and Wolpert of the isospectral problem for planar domains [4], [5] has aroused new interest in other isospectrality problems. Another problem of long standing is to find the largest n such that any positive definite quadratic form of rank n is determined by its representation numbers. Equivalently, what is the smallest dimension in which there exist two inequivalent lattices with the same theta series? We shall call such lattices *isospectral*. Witt [13], Kneser [6] and Kitaoka [7] found isospectral lattices in dimensions 16, 12 and 8 respectively. Milnor [8] pointed out the connection with the isospectral manifold problem.

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