Penny-Packing and Two-Dimensional Codes*

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ABSTRACT

We consider the problem of packing n equal circles (i.e. pennies) in the plane so as to minimize the second moment U about their centroid. These packings are also minimal-energy two-dimensional codes. Adding one penny at a time according to the greedy algorithm produces a unique sequence of packings for the first 75 pennies, and appears to produce optimal packings for infinitely many values of n. Several other conjectures are proposed, and a table is given of the best packings known for $n \leq 500$. For large n, $U \sim \sqrt{3}\ n^2/(4\pi)$.

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1. The Penny-Packing Problem

We wish to find the tightest packing in the plane of n equal circular disks, pennies for example, for $n = 1, 2, \ldots$. Groemer, Wegner and others have considered minimizing the area of the convex hull of the pennies [8], [15]. But for our applications it is more appropriate to minimize their second moment.

Suppose the pennies have diameter d, let $P_1, \ldots, P_n$ be their centers and $\bar{P} = n^{-1} \sum P_i$ their centroid. Then the problem is to choose points $P_1, \ldots, P_n$ so as to satisfy

$$\| P_i - P_j \| \geq d \quad , \quad i,j = 1, \ldots, n,\ i \neq j \ , \tag{1}$$

and so that the *second moment*

$$U = \frac{1}{d^2} \sum_{i=1}^{n} \| P_i - \bar{P} \|^2 \tag{2}$$

is minimized, where $\| \ \ \|$ is Euclidean distance. Let $U(n)$ be the minimal value of U (which can always be attained). A set of points $3 = \{P_1, \ldots, P_n\}$ satisfying (1) is called an n-point *packing*, and is *optimal* if it attains $U(n)$.

The second moment U is a reasonable parameter to minimize, for several reasons. For applications to signal design, when the points $P_1, \ldots, P_n$ form a two-dimensional *code* [3], [4], [7], U measures the energy in the code. Figure 1b for example shows what is conjectured to be the 16-point code with the smallest possible energy [4]. Furthermore the results look tightly packed (Figs. 1a-c), and when applied to packings in higher dimensions this criterion avoids the ‘‘sausage catastrophe’’ [5].

Penny packings also have applications in *cluster* chemistry [9], [14], quantization, and other areas [14]. Fig. 1c for example shows the 19 strands of wire in a cross-section of the cables used in the Brooklyn bridge [12]. The packing formed by pennies 1 to 37 in Fig. 4 occurs in the cross-section of an optical-fiber cable [11].

2. The Hexagonal Lattice

The familiar *hexagonal lattice* A_2 is spanned by $(1, 0)$ and $(1/2, \sqrt{3}/2)$. If we take $d = 1$ then any subset of A_2 is a packing.

Conjecture 1. For $n \neq 4$, every optimal packing is (apart from a magnification, rotation and translation) a subset of A_2.

The case $n = 4$ is exceptional, since all packings of the type shown in Fig. 2 have $U = 2$ (as long as each penny touches its two neighbors), and in general these are not subsets of A_2. Our investigations suggest that $n = 4$ is the only exception, although we cannot prove this.

Any point $P \in A_2$ can be written as

$$r(1, 0) + s\left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right] = \left[r + \frac{s}{2}, \frac{s\sqrt{3}}{2}\right]$$

for $r, s \in \mathbf{Z}$; we call $<r, s>$ oblique coordinates for P. There is a convenient formula for the second moment U of a subset $\{P_1, \ldots, P_n\} \subseteq A_2$. Let P_i have oblique coordinates $<r_i, s_i>$, and set $M_r = \sum r_i$, $M_s = \sum s_i$, $M_{rr} = \sum r_i^2$, $M_{rs} = \sum r_i s_i$, $M_{ss} = \sum s_i^2$. Then

$$U = (M_{rr} + M_{rs} + M_{ss}) - \frac{1}{n} (M_r^2 + M_r M_s + M_s^2) \ . \tag{3}$$

3. Constructions

We shall use three methods to construct packings.

(i) The greedy algorithm. A sequence of packings $3_1, 3_2, 3_3$,... is *produced by the greedy algorithm* if (a) 3_1 contains a single point, and (b) for $n = 2, 3, \ldots$,

$$3_n = 3_{n-1} \cup \{P_n\}$$

minimizes U over all choices of P_n satisfying (1). (There are many such sequences.)

(ii) Circular clusters in A_2 with a specified center. Let C be a fixed point of the plane, let t be a real number, and let

$$4 = \{P \in A_2 : \| P - C \| \le t\} \ .$$

In general the centroid of 4 will not coincide with C. If it does, we call 4 a *circular cluster* with center C.

The three most important cases occur when C is an element of A_2, e.g.

$$C_1 = <0, 0> \tag{4}$$

in oblique coordinates, C is midway between two neighboring elements of A_2, e.g.

$$C_2 = < \frac{1}{2} , 0> \ , \tag{5}$$

or C is at the center of three neighboring elements of A_2, e.g.

$$C_3 = < \frac{1}{3} , \frac{1}{3} > . \qquad (6)$$

Figures 1a and 1c are examples of clusters centered at C_1. Figure 5 below shows a cluster centered at C_2.

(iii) The iterative construction. The following iterative procedure produces an *n*-point packing with a small value of *U*. Choose a random initial point T_0. For $k = 1, 2, \ldots$ let 6 be the smallest circle around T_{k-1} that contains at least *n* points of A_2, let 5_k be obtained by discarding just enough (randomly chosen) points from the perimeter of 6 to leave *n* points, and let T_k be the centroid of 5_k. The sequence $5_1, 5_2, \ldots$ converges rapidly to an *n*-point circular cluster 5 in A_2.

The construction may now be repeated with a different choice for T_0, and after a number of tries the cluster with the smallest second moment is selected.

For example, when $n = 16$, if the initial point T_0 is close to the origin, the algorithm usually converges in one step to the cluster shown in Fig. 1b, which has centroid

$$C_{16} = < \frac{1}{4} , 0 > . \qquad (7)$$

4. Results

The most remarkable result, and our motivation for writing this note, is the following.

The greedy algorithm produces a unique packing for the first 75 pennies.

(‘‘Unique’’ means unique up to symmetry, of course. There are six ways to add a penny to Fig. 1a, for example, but all are equivalent under symmetries of A_2.)

There are two essentially distinct choices for the 76-th penny, both leading to packings with the same second moment. Whichever is chosen, the other must be chosen next, and the 77-point packing is unique (see Fig. 3). The packings now remain unique up to 313 pennies, when again there are two inequivalent choices. The 313-point packing is shown in Fig. 4, together with two inequivalent choices for the 314-th penny. The two paths merge again at the 318-th penny, and the next choice occurs at the 354-th penny (see Fig. 3).

The pennies in Fig. 4 are labeled sequentially, so in this figure one can see the n-point packings produced by the greedy algorithm for all $n \leq 313$.

Beyond 354 pennies we only followed one path of the greedy algorithm, which next splits at the 426-th penny.

Table I combines the best results produced by the three algorithms (using 100 random initial points for the iterative construction). Table 1a gives the smallest second moment found for all packings of $n \leq 100$ pennies. Table 1b gives a selection of the best packings found for $101 \leq n \leq 500$, including all the cases where the best packing is a cluster centered at C_1, C_2 or C_3, and some of the cases where the greedy algorithm is the best (often just giving the beginning and end of a run of successes for the greedy algorithm). %!PS-Adobe-2.0 %%Copyright: Copyright (c) 1993 AT&T, All Rights Reserved %%Version: 3.4 %%DocumentFonts: (atend) %%Pages: (atend) %%BoundingBox: (atend) %%EndComments /DpostDict 200 dict def DpostDict begin % % Copyright (c) 1993 AT&T, All Rights Reserved % % Version 3.4 prologue for troff files. %

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f (. . . ,)4 240(1 ,)1 130 2 3139 3588 t 12 I f (n)3569 3588 w 12 R f (,)3639 3588 w 12 I f (i)3719 3588 w 12 S f (271)3803 3588 w 12 I f (j)3929 3588 w 12 R f (,)1 1037 1 4003 3588 t (and so that the)3 698 1 720 3938 t 12 I f (second moment)1 742 1 1448 3938 t (U)2192 4344 w 12 S f (=)2338 4344 w 12 I f (d)2494 4446 w 9 R f (2)2569 4398 w 12 R f (1)2529 4272 w 12 S1 f (_ __)1 166 1 2476 4308 t 9 I f (i)2704 4470 w 9 S f (=)2751 4470 w 9 R f (1)2816 4470 w 17 S f (S)2732 4380 w 9 I f (n)2759 4234 w 12 S f (357 357)1 89 1 2900 4364 t 12 I f (P)3020 4344 w 9 I f (i)3108 4368 w 12 S f (-)3203 4344 w 12 I f (P)3329 4344 w 12 S1 f (_ _)1 61 1 3341 4233 t 12 S f (357 357)1 89 1 3432 4364 t 9 R f (2)3512 4296 w 12 R f ( )4900 4344 w (is minimized, where)2 1011 1 720 4796 t 12 S f (357)1 30(357)1 130(357 357)1 89 3 1777 4816 t 12 R f (Let)1 236(is Euclidean distance.)2 1077 2 2033 4796 t 12 I f (U)3393 4796 w 12 R f (489 4796 w 12 I f (n)3539 4796 w 12 R f () be the minimal value of)5 1298 1 3609 4796 t 12 I f (U)4954 4796 w 12 R f (set of points)3 655(A)1 158(attained).)1 493(ich can always be)3 1074 4 720 5076 t 12 UnivMath2 f (3)3142 5076 w 12 S f (=)3326 5076 w 12 I f ({ P)1 131 1 3452 5076 t 9 R f (1)3598 5100 w 12 R f (, . . . ,)4 270 1 3693 5076 t 12 I f (P)4023 5076 w 9 I f (n)4111 5100 w 12 I f (})4176 5076 w 12 R f (satisfying   is)2 774 1 4266 5076 t (called an)1 430 1 720 5356 t 12 I f (n)1180 5356 w 12 R f (-point)1240 5356 w 12 I f (packing)1558 5356 w 12 R f (, and is)2 344 1 1938 5356 t 12 I f (optimal)2312 5356 w 12 R f (if it attains)2 517 1 2710 5356 t 12 I f (U)3257 5356 w 12 R f (353 5356 w 12 I f (n)3403 5356 w 12 R f ().)3473 5356 w (moment)1 442(The second)1 565 2 900 5706 t 12 I f (U)1954 5706 w 12 R f (is a reasonable parameter to minimize, for several reasons.)8 2953 1 2087 5706 t (For applications to signal design, when the points)7 2428 1 720 5986 t 12 I f (P)3183 5986 w 9 R f (1)3271 6010 w 12 R f (, . . . ,)4 270 1 3366

5986 t 12 I f (P)3696 5986 w 9 I f (n)3784 6010 w 12 R f (form a two-dimensional)2 1166 1 3874 5986 t 12 I f (code)720 6266 w 12 R f ([3], [4], [7],)2 576 1 979 6266 t 12 I f (U)1588 6266 w 12 R f (1b for example shows what)5 1370(Figure)1 378(in the code.)3 599(measures the energy)2 986 4 1707 6266 t (Furthermore)1 662(possible energy [4].)3 984(is conjectured to be the 16-point code with the smallest)9 2674 3 720 6546 t (look tightly packed ffigs. 1a-c), and when applied to packings in higher)12 3803(the results)1 517 2 720 6826 t (dimensions this criterion avoids the ‘‘sausage catastrophe’’ [5].)7 3053 1 720 7106 t cleartomark showpage saveobj restore end %%PageBoundingBox: 61 64 514 702 %%EndPage: 2 2 %%Page: 2 3 %%PageBoundingBox: (atend) DpostDict begin /saveobj save def mark 3 pagesetup 12 R f (- 2 -)2 200 1 2780 560 t (also have applications in)4 1292(Penny packings)1 775 2 900 1120 t 12 I f (cluster)3016 1120 w 12 R f (chemistry [9], [14], quantization,)3 1647 1 3393 1120 t (1c for example shows the 19 strands of wire in a cross-section)12 3064(Fig.)1 255(and other areas [14].)3 1001 3 720 1400 t (packing formed by pennies 1 to 37 in)8 1830(The)1 247(the Brooklyn bridge [12].)4 1262(of the cables used in)4 981 4 720 1680 t (Fig. 4 occurs in the cross-section of an optical-256ber cable [11].)10 3017 1 720 1960 t %INFO[SECTION: LEVEL = 1, NUMBER = 2. , HEADING = The Hexagonal Lattice] 12 B f (Hexagonal Lattice)2 968(2. The)1 350 2 720 2380 t 12 R f (The familiar)1 603 1 900 2735 t 12 I f (hexagonal lattice A)2 945 1 1538 2735 t 9 R f (2)2498 2759 w 12 R f (1)1 70(and 285(0))1 140(,)1 40(1 136(is spanned by)2 664 6 2588 2735 t 12 I f (/)3933 2735 w 12 R f (2 ,)1 100 1 3977 2735 t 13 S f (326‘ ‘)1 152 1 4127 2735 t 12 R f (3)4209 2735 w 12 I f (/)4299 2735 w 12 R f (we take)2 411(If)1 146(2).)1 140 3 4343 2735 t 12 I f (d)720 3015 w 12 S f (=)840 3015 w 12 R f (1 then any subset of)4 961 1 966 3015 t 12 I f (A)1957 3015 w 9 R f (2)2045 3039 w 12 R f (is a packing.)2 604 1 2130 3015 t 12 B f (Conjecture 1)1 666 1 720 3435 t 12 R f (. For)1 261 1 1386 3435 t 12 I f (n)1681 3435 w 12 S f (271)1791 3435 w 12 R f (packing is ~art from a magni256cation, rotation)7 2340(4, every optimal)2 793 2 1907 3435 t (and translation) a subset of)4 1296 1 720 3715 t 12 I f (A)2046 3715 w 9 R f (2)2134 3739 w 12 R f (.)2189 3715 w (The case)1 434 1 900 4065 t 12 I f (n)1377 4065 w 12 S f (=)1497 4065 w 12 R f (4 is exceptional, since all packings of the type shown in Fig. 2 have)13 3417 1 1623 4065 t 12 I f (U)720 4345 w 12 S f (=)866 4345 w 12 R f (penny touches its two neighbors), and in general these are not)11 3139(2 long as each)4 909 2 992 4345 t (subsets of)1 483 1 720 4625 t 12 I f (A)1238 4625 w 9 R f (2)1326 4649 w 12 R f (investigations suggest that)3 1320(. Our)1 282 2 1381 4625 t 12 I f (n)3019 4625 w 12 S f (=)3139 4625 w 12 R f (4 is the only exception, although we)6 1775 1 3265 4625 t (cannot prove this.)2 858 1 720 4905 t (Any point)1 484 1 900 5255 t 12 I f (P)1414 5255 w 12 S f (316)1537 5255 w 12 I f (A)1672 5255 w 9 R f (2)1760 5279 w 12 R f (can be written as)3 810 1 1845 5255 t 12 I f (r)1721 5707 w 12 R f ( ))1 50(0)1 100(1 ,)2 150 3 1778 5707 t 12 S f (+)2148 5707 w 12 I f (s)2274 5707 w 12 S f (354)2341 5602 w (357)2341 5722 w (356)2341 5842 w 12 R f (2)2430 5811 w (1)2430 5655 w 12 S1 f (_ _)1 96 1 2412 5691 t 12 R f (,)2570 5727 w (2)2731 5811 w 13 S f (326‘ ‘)1 152 1 2680 5655 t 12 R f (3)2762 5655 w 12 S1 f (_ ___)1 198 1 2662 5691 t 12 S f (374)2872 5602 w (357)2872 5722 w (376)2872 5842 w (=)2991 5707 w (354)3127 5602 w (357)3127 5722 w (356)3127 5842 w 12 I f (r)3186 5727 w 12 S f (+)3293 5727 w 12 R f (2)3449 5811 w 12 I f (s)3456 5655 w 12 S1 f (_ _)1 96 1 3432 5691 t 12 R f (,)3590 5727 w (2)3794 5811 w 12 I f (s)3700 5655 w 13 S f (326‘ ‘)1 152

1 3787 5655 t 12 R f (3)3869 5655 w 12 S1 f (_ ____)1 285 1 3682 5691 t 12 S f (374)3979 5602 w (357)3979 5722 w (376)3979 5842 w 12 R f (for)720 6183 w 12 I f (r)896 6183 w 12 R f (,)983 6183 w 12 I f (s)1053 6183 w 12 S f (316)1150 6183 w 12 B f (Z)1285 6183 w 12 R f (; we call)2 419 1 1365 6183 t 12 S f (<)1820 6183 w 12 I f (r)1906 6183 w 12 R f (,)1993 6183 w 12 I f (s)2063 6183 w 12 S f (>)2140 6183 w 12 R f (oblique coordinates for)2 1127 1 2242 6183 t 12 I f (P)3405 6183 w 12 R f (formula)1 418(is a convenient)3 769(. There)1 375 3 3478 6183 t (for the second moment)3 1201 1 720 6463 t 12 I f (U)1982 6463 w 12 R f (of a subset)2 576 1 2129 6463 t 12 I f ({ P)1 131 1 2766 6463 t 9 R f (1)2912 6487 w 12 R f (, . . . ,)4 270 1 3007 6463 t 12 I f (P)3337 6463 w 9 I f (n)3425 6487 w 12 I f (})3490 6463 w 12 S f (315)3588 6463 w 12 I f (A)3723 6463 w 9 R f (2)3811 6487 w 12 R f (. Let)1 281 1 3866 6463 t 12 I f (P)4208 6463 w 9 I f (i)4296 6487 w 12 R f (have oblique)1 648 1 4392 6463 t (coordinates)720 6743 w 12 S f (<)1650 6743 w 12 I f (r)1736 6743 w 9 I f (i)1798 6767 w 12 R f (,)1883 6743 w 12 I f (s)1963 6743 w 9 I f (i)2025 6767 w 12 S f (>)2080 6743 w 12 R f (, and set)2 1091 1 2146 6743 t 12 I f (M)3614 6743 w 9 I f (r)3729 6767 w 12 S f (=)3834 6743 w 17 S f (S)3980 6779 w 12 I f (r)4150 6743 w 9 I f (i)4212 6767 w 12 R f (,)4297 6743 w 12 I f (M)4377 6743 w 9 I f (s)4492 6767 w 12 S f (=)4597 6743 w 17 S f (S)4743 6779 w 12 I f (s)4913 6743 w 9 I f (i)4975 6767 w 12 R f (,)5010 6743 w 12 I f (M)720 7035 w 9 I f (rr)835 7059 w 12 S f (=)975 7035 w 17 S f (S)1121 7071 w 12 I f (r)1291 7035 w 9 I f (i)1345 7059 w 9 R f (2)1345 6988 w 12 R f (,)1450 7035 w 12 I f (M)1530 7035 w 9 I f (rs)1645 7059 w 12 S f (=)1785 7035 w 17 S f (S)1931 7071 w 12 I f (r)2101 7035 w 9 I f (i)2163 7059 w 12 I f (s)2238 7035 w 9 I f (i)2300 7059 w 12 R f (,)2385 7035 w 12 I f (M)2465 7035 w 9 I f (ss)2580 7059 w 12 S f (=)2720 7035

w 17 S f (S)2866 7071 w 12 I f (s)3036 7035 w 9 I f (i)3090 7059 w 9 R f (2)3090 6988 w 12 R f (. Then)1 336 1 3145 7035 t cleartomark showpage saveobj restore end %%PageBoundingBox: 61 65 514 758 %%EndPage: 2 3 %%Page: 3 4 %%PageBoundingBox: (atend) DpostDict begin /saveobj save def mark 4 pagesetup 12 R f (- 3 -)2 200 1 2780 560 t 12 I f (U)1414 1168 w 12 S f (=)1560 1168 w 12 R f (686 1168 w 12 I f (M)1736 1168 w 9 I f (rr)1851 1192 w 12 S f (+)1991 1168 w 12 I f (M)2117 1168 w 9 I f (rs)2232 1192 w 12 S f (+)2372 1168 w 12 I f (M)2498 1168 w 9 I f (ss)2613 1192 w 12 R f ())2703 1168 w 12 S f (-)2813 1168 w 12 I f (n)2969 1252 w 12 R f (1)2969 1096 w 12 S1 f (_ _)1 96 1 2951 1132 t 12 R f (099 1168 w 12 I f (M)3149 1168 w 9 I f (r)3256 1192 w 9 R f (2)3256 1121 w 12 S f (+)3371 1168 w 12 I f (M)3497 1168 w 9 I f (r)3612 1192 w 12 I f (M)3697 1168 w 9 I f (s)3812 1192 w 12 S f (+)3917 1168 w 12 I f (M)4043 1168 w 9 I f (s)4150 1192 w 9 R f (2)4150 1121 w 12 R f ())4215 1168 w 12 I f (.)4315 1168 w 12 R f ( )4900 1168 w %INFO[SECTION: LEVEL = 1, NUMBER = 3. , HEADING = Constructions] 12 B f (3. Constructions)1 871 1 720 1718 t 12 R f (We shall use three methods to construct packings.)7 2404 1 900 2068 t 12 B f (algorithm)1 551(The greedy)2 744 2 720 2488 t 12 R f (sequence of packings)3 1095(. A)1 189 2 2015 2488 t 12 UnivMath2 f (3)3342 2488 w 9 R f (1)3481 2512 w 12 R f (,)3546 2488 w 12 UnivMath2 f (3)3626 2488 w 9 R f (2)3765 2512 w 12 R f (,)3830 2488 w 12 UnivMath2 f (3)3910 2488 w 9 R f (3)4049 2512 w 12 R f (,... is)1 244 1 4144 2488 t 12 I f (produced by)1 609 1 4431 2488 t (the greedy algorithm)2 1008 1 720 2768 t 12 R f (if)1 237 1 1758 2768 t 12 UnivMath2 f (3)2025 2768 w 9 R f (1)2164 2792 w 12 R f (contains a single point, and for)6 1653 1 2249 2768 t 12 I f (n)3932 2768 w 12 S f (=)4052 2768 w 12 R f (,)1 40(3)1 100(2 ,)1 100 3 4178 2768 t 12 I f

(...)4458 2768 w 12 R f (,)4548 2768 w 12 UnivMath2 f (3)2301 3118 w 9 I f (n)2440 3142 w 12 S f (=)2555 3118 w 12 UnivMath2 f (3)2681 3118 w 9 I f (n)2820 3142 w 9 S f (-)2887 3142 w 9 R f (1)2952 3142 w 12 S f (310)3057 3118 w 12 I f ({ P)1 131 1 3199 3118 t 9 I f (n)3345 3142 w 12 I f (})3410 3118 w 12 R f (minimizes)720 3468 w 12 I f (U)1253 3468 w 12 R f (over all choices of)3 884 1 1369 3468 t 12 I f (P)2283 3468 w 9 I f (n)2371 3492 w 12 R f (are many such sequences.))4 1309(ere)1 379(satisfying .)1 669 3 2456 3468 t 12 B f ( ) Circular clusters in)3 1179 1 720 3888 t 12 I f (A)1931 3888 w 9 R f (2)2019 3912 w 12 B f (with a speci256ed center)3 1146 1 2106 3888 t 12 R f (. Let)1 253 1 3252 3888 t 12 I f (C)3538 3888 w 12 R f (be a 256xed point of the plane,)6 1389 1 3651 3888 t (let)720 4168 w 12 I f (t)871 4168 w 12 R f (be a real number, and let)5 1187 1 935 4168 t 12 UnivMath2 f (4)2002 4518 w 12 S f (=)2172 4518 w 12 I f ({ P)1 131 1 2298 4518 t 12 S f (316)2479 4518 w 12 I f (A)2614 4518 w 9 R f (2)2702 4542 w 12 R f (:)2797 4518 w 12 S f (357 357)1 89 1 2871 4538 t 12 I f (P)2991 4518 w 12 S f (-)3114 4518 w 12 I f (C)3230 4518 w 12 S f (357 357)1 89 1 3340 4538 t (243)3470 4518 w 12 I f (.)1 80(t })1 92 2 3586 4518 t 12 R f (In general the centroid of)4 1258 1 720 4868 t 12 UnivMath2 f (4)2019 4868 w 12 R f (will not coincide with)3 1086 1 2170 4868 t 12 I f (C)3297 4868 w 12 R f (it does, we call)4 795(. If)1 181 2 3377 4868 t 12 UnivMath2 f (4)4405 4868 w 12 R f (a)4557 4868 w 12 I f (circular)4652 4868 w (cluster)720 5148 w 12 R f (with center)1 537 1 1078 5148 t 12 I f (C)1645 5148 w 12 R f (.)1725 5148 w (The three most important cases occur when)6 2088 1 900 5498 t 12 I f (C)3018 5498 w 12 R f (is an element of)3 765 1 3128 5498 t 12 I f (A)3923 5498 w 9 R f (2)4011 5522 w 12 R f (, e.g.)1 233 1 4066 5498 t 12 I f (C)2521 5848 w 9 R f (1)2616 5872 w 12 S f (= <)1 202 1 2731 5848 t 12 R f (0)1 100(0 ,)1 100 2 2953 5848 t

12 S f (>)3173 5848 w 12 R f ( )4900 5848 w (in oblique coordinates,)2 1099 1 720 6198 t 12 I f (C)1849 6198 w 12 R f (is midway between two neighboring elements of)6 2336 1 1959 6198 t 12 I f (A)4325 6198 w 9 R f (2)4413 6222 w 12 R f (, e.g.)1 233 1 4468 6198 t 12 I f (C)2436 6596 w 9 R f (2)2531 6620 w 12 S f (= <)1 202 1 2646 6596 t 12 R f (2)2898 6680 w (1)2898 6524 w 12 S1 f (_ _)1 96 1 2880 6560 t 12 R f (, 0)1 130 1 3028 6596 t 12 S f (>)3178 6596 w 12 R f (,)1 1746 1 3294 6596 t (or)720 7006 w 12 I f (C)850 7006 w 12 R f (is at the center of three neighboring elements of)8 2297 1 960 7006 t 12 I f (A)3287 7006 w 9 R f (2)3375 7030 w 12 R f (, e.g.)1 233 1 3430 7006 t cleartomark showpage saveobj restore end %%PageBoundingBox: 61 74 514 758 %%EndPage: 3 4 %%Page: 4 5 %%PageBoundingBox: (atend) DpostDict begin /saveobj save def mark 5 pagesetup 12 R f (- 4 -)2 200 1 2780 560 t 12 I f (C)2401 1168 w 9 R f (3)2496 1192 w 12 S f (= <)1 202 1 2611 1168 t 12 R f (3)2863 1252 w (1)2863 1096 w 12 S1 f (_ _)1 96 1 2845 1132 t 12 R f (,)2993 1168 w (3)3093 1252 w (1)3093 1096 w 12 S1 f (_ _)1 96 1 3075 1132 t 12 S f (>)3203 1168 w 12 I f (.)3329 1168 w 12 R f ( )4900 1168 w (Figures 1a and 1c are examples of clusters centered at)9 2780 1 720 1578 t 12 I f (C)3551 1578 w 9 R f (1)3646 1602 w 12 R f (5 below shows a)4 914(. Figure)1 425 2 3701 1578 t (cluster centered at)2 874 1 720 1858 t 12 I f (C)1624 1858 w 9 R f (2)1719 1882 w 12 R f (.)1774 1858 w 12 B f (construction)1 680(i) The iterative)2 885 2 720 2278 t 12 R f (following iterative procedure produces an)5 2083(. The)1 285 2 2285 2278 t 12 I f (n)4692 2278 w 12 R f (-point)4752 2278 w (packing with a small value of)5 1449 1 720 2558 t 12 I f (U)2205 2558 w 12 R f (a random initial point)4 1095(. Choose)1 456 2 2291 2558 t 12 I f (T)3878 2558 w 9 R f (0)3960 2582 w 12 R f (. For)1 263 1 4015 2558 t 12 I f (k)4314 2558 w 12 S f (=)4427 2558 w 12 R f (

let)1 157(,...)1 130(2)1 100(1 ,)1 100 4 4553 2558 t 12 UnivMath2 f (6)720 2838 w 12 R f (around)1 376(be the smallest circle)3 1049 2 878 2838 t 12 I f (T)2346 2838 w 9 I f (k)2428 2862 w 9 S f (-)2490 2862 w 9 R f (1)2555 2862 w 12 R f (that contains at least)3 1019 1 2653 2838 t 12 I f (n)3715 2838 w 12 R f (points of)1 438 1 3818 2838 t 12 I f (A)4299 2838 w 9 R f (2)4387 2862 w 12 R f (, let)1 194 1 4442 2838 t 12 UnivMath2 f (5)4689 2838 w 9 I f (k)4834 2862 w 12 R f (be)4927 2838 w (from the perimeter of)4 1074(obtained by discarding just enough ndomly chosen) points)7 2970 2 720 3118 t 12 UnivMath2 f (6)4797 3118 w 12 R f (to)4946 3118 w (leave)720 3398 w 12 I f (n)1007 3398 w 12 R f (points, and let)2 687 1 1101 3398 t 12 I f (T)1823 3398 w 9 I f (k)1905 3422 w 12 R f (be the centroid of)3 859 1 1990 3398 t 12 UnivMath2 f (5)2884 3398 w 9 I f (k)3029 3422 w 12 R f (sequence)1 474(. The)1 281 2 3079 3398 t 12 UnivMath2 f (5)3869 3398 w 9 R f (1)4014 3422 w 12 R f (,)4079 3398 w 12 UnivMath2 f (5)4159 3398 w 9 R f (2)4304 3422 w 12 R f (,... converges)1 641 1 4399 3398 t (rapidly to an)2 608 1 720 3678 t 12 I f (n)1358 3678 w 12 R f (-point circular cluster)2 1036 1 1418 3678 t 12 UnivMath2 f (5)2484 3678 w 12 R f (in)2644 3678 w 12 I f (A)2768 3678 w 9 R f (2)2856 3702 w 12 R f (.)2911 3678 w (a different choice for)4 1116(The construction may now be repeated with)6 2215 2 900 4028 t 12 I f (T)4280 4028 w 9 R f (0)4362 4052 w 12 R f (, and after a)3 623 1 4417 4028 t (number of tries the cluster with the smallest second moment is selected.)11 3456 1 720 4308 t (For example, when)2 927 1 900 4658 t 12 I f (n)1859 4658 w 12 S f (=)1979 4658 w 12 R f (point)1 281(16, if the initial)3 750 2 2105 4658 t 12 I f (T)3169 4658 w 9 R f (0)3251 4682 w 12 R f (is close to the origin, the algorithm)6 1701 1 3339 4658 t (usually converges in one step to the cluster shown in Fig. 1b, which has centroid)14 3872 1 720

4938 t 12 I f (C)2408 5336 w 9 R f (16)2503 5360 w 12 S f (= <)1 202 1 2663 5336 t 12 R f (4)2915 5420 w (1)2915 5264 w 12 S1 f (_ _)1 96 1 2897 5300 t 12 R f (, 0)1 130 1 3045 5336 t 12 S f (>)3195 5336 w 12 I f (.)3321 5336 w 12 R f ( )4900 5336 w %INFO[SECTION: LEVEL = 1, NUMBER = 4. , HEADING = Results] 12 B f (4. Results)1 524 1 720 5886 t 12 R f (The most remarkable result, and our motivation for writing this note, is the following.)13 4131 1 900 6236 t (The greedy algorithm produces a unique packing for the 256rst 75 pennies.)11 3496 1 900 6586 t (are six ways to add a)6 1123(There)1 353(“Unique’’ means unique up to symmetry, of course.)7 2664 3 900 6936 t (penny to Fig. 1a, for example, but all are equivalent under symmetries of)12 3509 1 720 7216 t 12 I f (A)4259 7216 w 9 R f (2)4347 7240 w 12 R f (.))4402 7216 w cleartomark showpage saveobj restore end %%PageBoundingBox: 61 53 514 758 %%EndPage: 4 5 %%Page: 5 6 %%PageBoundingBox: (atend) DpostDict begin /saveobj save def mark 6 pagesetup 12 R f (- 5 -)2 200 1 2780 560 t (penny, both leading to)4 1249(There are two essentially distinct choices for the 76-th)8 2891 2 900 1120 t (must be chosen)3 783(is chosen, the other)4 978(Whichever)1 591(packings with the same second moment.)5 1968 4 720 1400 t (packings now remain unique)4 1450(The)1 255(packing is unique e Fig. 3).)6 1536(next, and the 77-point)3 1079 4 720 1680 t (313-point packing)2 908(The)1 246(there are two inequivalent choices.)5 1701(up to 313 pennies, when again)5 1465 4 720 1960 t (two)1 213(The)1 249(inequivalent choices for the 314-th penny.)6 2077(is shown in Fig. 4, together with two)7 1781 4 720 2240 t (at the 354-th penny)4 1009(paths merge again at the 318-th penny, and the next choice occurs)11 3311 2 720 2520 t (e Fig. 3).)2 574 1 720 2800 t (Fig. 4 are labeled sequentially, so in this 256gure one can see the)13 3054(The pennies in)2 707 2 900 3150

t 12 I f (n)4692 3150 w 12 R f (-point)4752 3150 w (packings produced by the greedy algorithm for all)7 2406 1 720 3430 t 12 I f (n)3156 3430 w 12 S f (243)3266 3430 w 12 R f (313.)3382 3430 w (path of the greedy algorithm, which next)7 2045(Beyond 354 pennies we only followed one)6 2095 2 900 3780 t (splits at the 426-th penny.)4 1247 1 720 4060 t (ing 100 random)3 941(Table I combines the best results produced by the three algorithms)10 3199 2 900 4410 t (second moment)2 802(1a gives the smallest)4 1062(Table)1 341(initial points for the iterative construction).)5 2115 4 720 4690 t (found for all packings of)4 1308 1 720 4970 t 12 I f (n)2089 4970 w 12 S f (243)2199 4970 w 12 R f (1b gives a selection of the best)7 1723(Table)1 364(100 pennies.)1 638 3 2315 4970 t (packings found for 101)3 1159 1 720 5250 t 12 S f (243)1929 5250 w 12 I f (n)2045 5250 w 12 S f (243)2155 5250 w 12 R f (best packing is a)4 880(500, including all the cases where the)6 1889 2 2271 5250 t (cluster centered at)2 892 1 720 5530 t 12 I f (C)1651 5530 w 9 R f (1)1746 5554 w 12 R f (,)1811 5530 w 12 I f (C)1891 5530 w 9 R f (2)1986 5554 w 12 R f (or)2080 5530 w 12 I f (C)2219 5530 w 9 R f (3)2314 5554 w 12 R f (some of the cases where the greedy algorithm is)9 2429(, and)1 242 2 2369 5530 t (for the greedy)3 751(the best ten just giving the beginning and end of a run of successes)13 3569 2 720 5810 t (algorithm).)720 6090 w cleartomark showpage saveobj restore end %%PageBoundingBox: 61 166 514 758 %%EndPage: 5 6 %%Trailer DpostDict begin done end %%Pages: 6 %%DocumentFonts: Times-Bold Times-Italic Times-Roman Times-Roman Symbol Universal-MathTwo