Descending Dungeons

(Proposed problem—with solution—for the American Mathematical Monthly)

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Consider the sequences

$$10,\ 10_{11},\ 10_{11_{12}},\ 10_{11_{12_{13}}},\ \ldots,$$
$$10,\ 11_{10},\ 12_{11_{10}},\ 13_{12_{11_{10}}},\ \ldots,$$

where a_b (or more conveniently a_b) denotes "a converted from decimal to base b." That is, if $a = \sum_{i=0}^{k} a_i 10^i$ (with $a_i \in [0,9]$) then $a\_b = \sum_{i=0}^{k} a_i b^i$. Since the iterated subscripts can be grouped from the bottom upwards or the top downwards, there are really four sequences:

(i) 10, 10_11, 10_(11_12), 10_(11_(12_13)), . . . ,

(ii) 10, 10_11, (10_11)_12, ((10_11)_12)_13, . . . ,

(iii) 10, 11_10, 12_(11_10), 13_(12_(11_10)), . . . ,

(iv) 10, 11_10, (12_11)_10, ((13_12)_11)_10, ...

Show that if $s(n)$ denotes the nth term in any one of these sequences (indexed by $n = 10, 11, \ldots$), then $\log\log s(n) \sim n \log\log n$, where the logarithms are to base 10.

(Sequences (i)–(iv) are now entries A121263, A121265, A121295, A121296 in the *On-Line Encyclopedia of Integer Sequences*.)