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Some Lattices Obtained from Riemann Surfaces

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Abstract. We compute the period matrices for the family of Riemann surfaces defined by $w^2 = z^{2n} - 1$ and examine the resulting lattices. Rather surprisingly, the case $n = 4$ yields the recently discovered "mean centered cuboidal" lattice, which is known to be both the densest isodual lattice packing in three dimensions and the best covering.

1. Introduction

The period matrix of a compact Riemann surface of genus g determines a real $2g$-dimensional lattice. Buser and Sarnak [**A**] have shown that from a sphere packing point of view these lattices are somewhat disappointing: for large g their density is much worse than the Minkowski bound, neither the root lattices E_6, E_8 nor the Leech lattice can be obtained, and so on. Our investigations were motivated by the desire to see what packings could be obtained in low dimensions. We focused on hyperelliptic Riemann surfaces, and this report summarizes our findings. As mentioned in the abstract, the results were a surprise. (For some similar computations on other Riemann surfaces, see [**E**].)

2. The Period Matrix

Let M be the hyperelliptic Riemann surface of genus g defined by the equation $w^2 = z^{2g+2} - 1$; M is a double cover of the Riemann sphere, branched at powers of $\zeta = e^{\frac{\pi i}{g+1}}$. The line segment L_j from ζ^j to ζ^{j+1} has two lifts to M, which, after a suitable choice of branch cuts, we can denote by L_j^+ and L_j^-. Then the closed loops

$$a_j = L_{2j}^+ - L_{2j}^-, \quad 0 \le j < g \ ,$$

$$b_j = \sum_{k=j}^{g-1} (L_{2k+1}^+ - L_{2k+1}^-), \quad 0 \le j < g$$

determine a canonical homology basis for M, i.e. a homology basis with intersection matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

As a basis for the space of holomorphic differentials on M we pick

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$$\omega_i = \frac{c_i z^i dz}{w}, \quad 0 \le i < j, \quad c_i \in \mathbb{C}^\star \ ,$$

choosing the constants c_i so that $2\int_{L_0^+} \omega_i = 1$. The period matrix for M is then the $g \times 2g$ matrix $(I \quad Z)$, where

$$Z = A^{-1}B, \quad (A_{ij}) = \int_{a_j} \omega_i, \quad (B_{ij}) = \int_{b_j} \omega_i \ .$$

Ordinarily, such integrals are extremely difficult to compute, but in our case the symmetry of the problem allows us to obtain Z purely in terms of ζ:

$$\int_{a_j} \omega_i \ = \ 2\int_{L_{2j}^+} \omega_i \ = \ \zeta^{2j(i+1)} \cdot 2\int_{L_0^+} \omega_i \ = \ \zeta^{2j(i+1)},$$

$$\int_{b_j} \omega_i \ = \ \sum_{k=j}^{g-1} \zeta^{(2k+1)(i+1)}.$$

Like all period matrices, Z is symmetric and positive definite. A routine computation shows that in our case Z is purely imaginary, and $Y = \operatorname{Im} Z$ has determinant 1.

3. The Period Lattice

The period matrix defines a lattice in the g-dimensional complex vector space $H^0(K_M)^\star$ dual to the space of holomorphic differentials on M. To this lattice is associated a positive definite Hermitian form, its *polarization*, which we can use to define an inner product on $H^0(K_M)^\star$ considered as a $2g$-dimensional real vector space. The Gram matrix of the period lattice is then

$$\begin{pmatrix} Y^{-1} & 0 \\ 0 & Y \end{pmatrix}.$$

(For details of this construction, see Appendix 0 of [**A**].) Thus the period lattice of M is a direct sum of two g-dimensional lattices. In fact, the two lattices are congruent:

THEOREM. *The lattice with Gram matrix Y is isodual (geometrically congruent to its dual). It has symmetry group $\pm\mathbf{D}_{2g+2}$ of order $4g+4$, and kissing number $2g+2$.*

PROOF. The automorphism of M sending (z, w) to $(\zeta z, w)$ acts on the homology basis $\{a_0, \ldots, a_{g-1}, b_0, \ldots, b_{g-1}\}$ via the matrix

$$\begin{pmatrix} 0 & -{}^tC^{-1} \\ C & 0 \end{pmatrix}, \quad \text{where} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

is integral and unimodular. The induced action on the period lattice preserves the polarization, and therefore also the Gram matrix. Hence ${}^tCYC = Y^{-1}$, and Y is isodual.

By Torelli's theorem, the group of automorphisms of any hyperelliptic Riemann surface is naturally isomorphic to the group of automorphisms of its polarized Jacobian, i.e. those symmetries of the period lattice that are given by symplectic matrices [**D**]. The automorphism group of our surface M is $(\pm 1) \times \mathbf{D}_{4g+4}$, corresponding to the hyperelliptic involution and the dihedral symmetries of the branch points of M. The group contains an index 2 subgroup of elements that do not interchange the two copies of Y in the period lattice; this subgroup is the full symmetry group of Y, and is easily seen to be $(\pm 1) \times \mathbf{D}_{2g+2}$. □

4. Examples

What isodual lattices Y do we obtain for low genera?

$g = 1$.

$$Y = (1) \qquad \text{(of course)}$$

$g = 2$.

$$Y = \frac{1}{\sqrt{3}} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

This is the hexagonal lattice A_2, scaled to have determinant 1.

$g = 3$.

$$Y = \frac{1}{2} \begin{pmatrix} 1+\sqrt{2} & \sqrt{2} & 1 \\ \sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 1 & \sqrt{2} & 1+\sqrt{2} \end{pmatrix}$$

This is a surprise. This lattice is the recently discovered "mean centered cuboidal" (or m.c.c.) lattice, which is in a sense the geometrical mean of the face centered cubic lattice and the body centered cubic lattice [**C**]. It is both the densest packing and thinnest covering among three-dimensional isodual lattices.

$g = 4$. Unfortunately, that property does not generalize to higher genera: the best isodual packing and covering in four dimensions is the root lattice D_4 [**C**], whereas the Gram matrix of our lattice is

$$Y = \begin{pmatrix} 2 & a-3 & 2-a & 2-a \\ a-3 & 2 & a-3 & 2-a \\ 2-a & a-3 & 2 & a-3 \\ 2-a & 2-a & a-3 & 2 \end{pmatrix},$$

where $a = \sqrt{5}$. This has minimal norm 2, determinant $125(9-4\sqrt{5})$, kissing number 10, automorphism group $\pm\mathbf{D}_{10}$, and the rather low center density of 0.09472 (cf. [**B**], Table 1.2). On the other hand replacing a in the Gram matrix by respectively 2 and 5/2 produces the lattices A_4 and A_4^*, so our lattice can be regarded as their geometric mean.

For higher genera these lattices do not seem so interesting, and we omit further details.

References

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