Graphical Enumeration and Stained Glass Windows, 1: Rectangular Grids

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DEDICATED TO THE MEMORY OF RONALD LEWIS GRAHAM (1935–2020)

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Abstract

A survey of enumeration problems arising from the study of graphs formed when the edges of a polygon are marked with evenly spaced points and every pair of points is joined by a line. A few of these problems have been solved, a classical example being the the graph K_n formed when all pairs of vertices of a regular n-gon are joined by chords, which was analyzed by Poonen and Rubinstein in 1998. Most of these problems are unsolved, however, and this two-part article provides data from a number of such problems as well as colored illustrations, which are often reminiscent of stained glass windows. The polygons considered include rectangles, hollow rectangles (or frames), triangles, pentagons, pentagrams, crosses, etc., as well as figures formed by drawing semicircles joining equally-spaced points on a line.

This first part discusses rectangular grids. The $1 \times n$ grids, or equally the graphs $K_{n+1,n+1}$, were studied by Legendre and Griffiths, and here we investigate the number of cells with a given number of edges and the number of nodes with a given degree. We have only partial results for the $m \times n$ rectangles, including upper bounds on the numbers of nodes and cells.

1 Introduction.

In 1998 Poonen and Rubinstein [17] (see also [22]) solved the problem of finding the numbers of intersection points and cells in a regular drawing of the complete graph K_n, and in 2009-2010 Legendre [10] and Griffiths [7] solved a similar problem for the complete bipartite graph $K_{n,n}$. Stated another way, [17] analyzes the graph formed by joining all pairs of vertices of a regular n-gon, while [10, 7] analyze the graph formed by taking a row of $n-1$ identical squares and drawing lines between every pair of boundary nodes.

One motivation for the present work was to see if these investigations could be extended to graphs formed from other structures, such as an $m \times n$ array of identical squares. Take a rectangle of size $m \times n$, and place $m-1$ equally spaced points on the two vertical sides, and $n-1$ equally spaced points on the two horizontal sides, and then draw lines between every pair of the $2(m+n)$ boundary points. The resulting planar graph, which we denote by $BC(m, n)$, is the main subject of Part 1 of this paper.

Although we have not been very successful in analyzing these graphs, we have collected a great deal of data, which has been entered into various sequences in the *On-Line Encyclopedia of Integer Sequences* [14].

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In Part 2 [3], we continue this work by considering other structures such as hollow squares (or "frames"), triangles, pentagons, hexagons, pentagrams, etc., as well as figures formed by drawing semicircles joining equally-spaced points on an interval.

We were also motivated by memories of stained glass windows seen in the great Gothic cathedrals of Northern Europe. In 2019 we made a colored drawing of K_{23} (Fig. 1) which was reminiscent of a rose window, and we were curious to see what colored versions of other graphs would look like. Informally, our philosophy has been, if we can't solve it, make art. We make no great claims for artistic merit, but the images are certainly colorful.

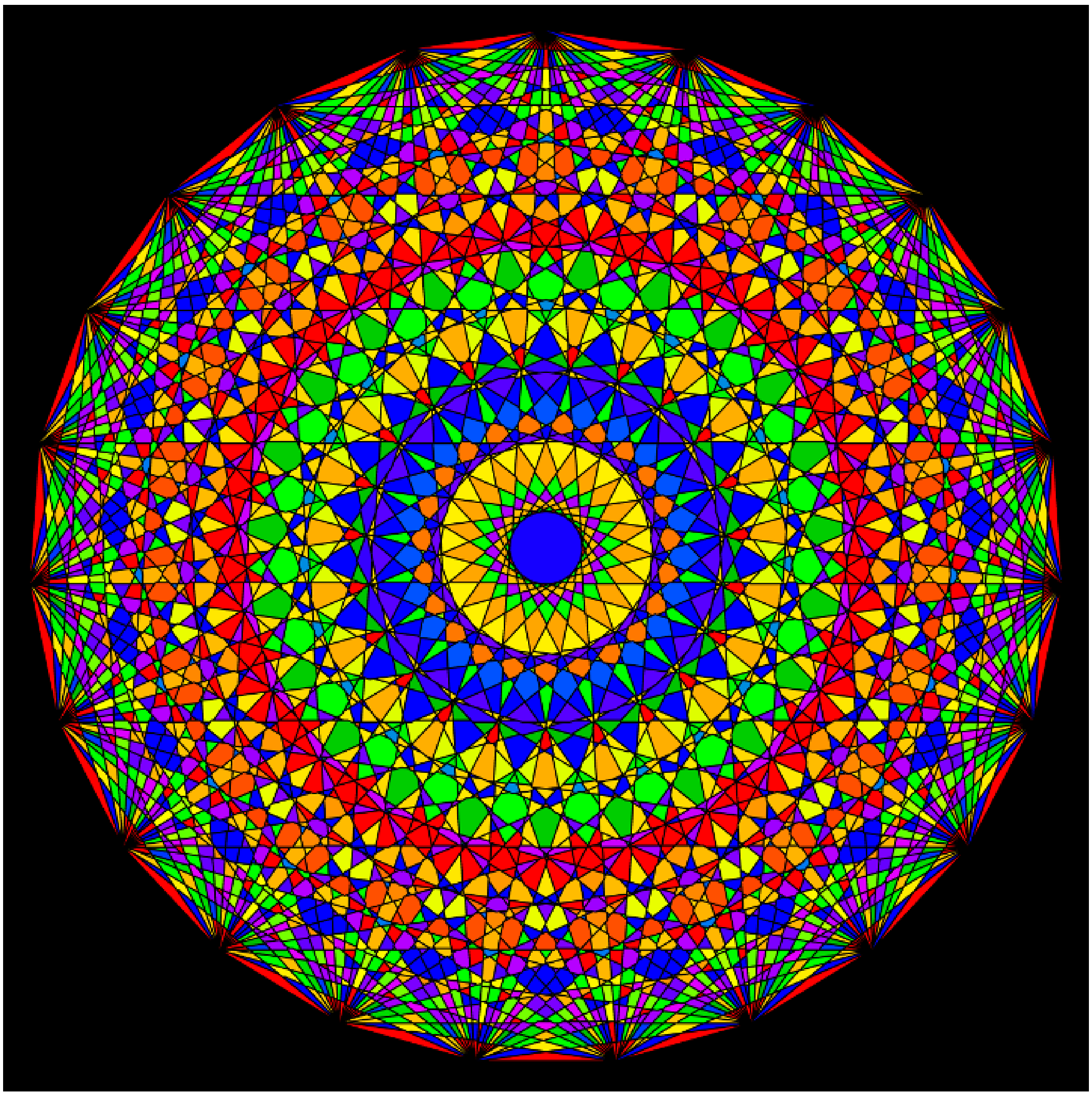

Figure 1: Colored drawing of complete graph K_{23} (see §10.3 for coloring algorithm). Entry A007678 in [14] has many similar images (which are also of higher quality).

Space limitations have restricted the number and quality of the images that we could include here. The corresponding entries in [14] (A007678[2] in the case of Fig. 1) contain a large number of other images, with better resolution. We are especially fond of the three images of K_{41} in A007678, and in A331452 the reader should not miss the images labeled $T(10,2)$, $T(6,6)$, $T(7,7)$, which are drawings of the graphs $BC(10,2)$, $BC(6,6)$, and $BC(7,7)$ discussed below.

This paper is arranged as follows. The last section of this Introduction establishes the notation we will use, especially the terms *nodes*, *chords*, and *cells*, and provides some examples. Section 2 deals with the graphs $BC(1,n)$ (or equivalently $BC(n,1)$), where the underlying polygon is a rectangle of size $1 \times n$ (or $n \times 1$). Theorem 2.1 gives Legendre and Griffiths's enumeration of the nodes and cells in $BC(1,n)$. In 2019, Max Alekseyev (personal communication; see also A306302) pointed out that that the Legendre-Griffiths results are essentially the same as results that he and his coauthors obtained in connection with the enumeration of two-dimensional threshold functions [1, 2]. The family of isosceles triangle graphs $IT(n)$ (Section 3) provides a bridge between the graphs $BC(1,n)$ and two-dimensional threshold function. Alekseyev also mentioned that their work implies a result that was apparently overlooked in the Legendre and Griffiths papers: the cells in $BC(1,n)$ are always triangles or quadrilaterals. See Theorem 3.1. The proof of this fact in [2] depends on a theorem about teaching sets for threshold function [19, 26]. We feel that such a elementary property should have a purely geometrical proof, although no such proof is presently known. We state this question as Open Problem 3.2.

One possible attack on this problem is to study the distribution of cells in each of the n squares of $BC(1,n)$—see Tables 2, 3, 4. The *gfun* Maple program [18] suggests a form for the generating functions of the columns of these tables, but so far this is only a conjecture.

We next consider the number of interior nodes in $BC(1,n)$ where c chords meet (Table 5). The number of simple nodes, where just two chords cross, is of the greatest interest, since these seem to dominate. But even though we have calculated 500 terms of this sequence (Table 6, A334701) we have been unable to find a formula or recurrence (Open Problem 5.2). There have been several similar occasions during this project when we have regretted not having an oracle that would take a few hundred terms of a simple, well-defined sequence and suggest some kind of formula.[3]

The graph $BC(1,n)$ starts from a $1 \times n$ rectangle. If we start from an $m \times n$ rectangle, with m and $n > 1$, there are actually three natural ways to define a graph, which we will denote by $BC(m,n)$, $AC(m,n)$, and $LC(m,n)$. These are the subjects of Sections 6, 8, and 9, respectively. For these families we have plenty of data and pictures, but not many results. In Section 6 we conjecture that the cells in $BC(2,n)$ have at most eight sides, and for $n \ge 19$, at most six sides (Conjecture 6.2). Our main result concerning $BC(m,n)$ is an upper bound on the numbers of nodes and cells in $BC(m,n)$, presented in §7, which appears to be reasonably close to the true values.

The final section (§10) describes the algorithms that were used to color the graphs.

Terminology. The graphs that we study are usually constructed by starting with a *polygon* P drawn in the plane, having *sides* and *vertices*. We then subdivide the sides by dividing them into some number of equal parts. To divide a side into k equal parts, we insert $k-1$ equally spaced *nodes* along that side. The side then contains the two end-vertices and the $k-1$ internal nodes. We say that the side has been *k-reticulated*. A *chord* in P is a finite line-segment joining a pair of vertices or nodes. A chord is undirected, does not extend outside P, and has specified end points.

Given a polygon P, we construct a planar graph G by drawing chords according to some specified rule. For

[2]Six-digit numbers prefixed by A refer to entries in the On-Line Encyclopedia of Integer Sequences [14].

[3]The oracle might compare the sequence with a shifted version of each of the 300000 entries in [14], and ask Bruno Salvy and Paul Zimmermann's program *gfun*, or Harm Derksen's program *guesss*, or Christian Krattenthaler's program *Rate*, or one of the other programs used by *Superseeker* [20] if there is a formula for the difference.

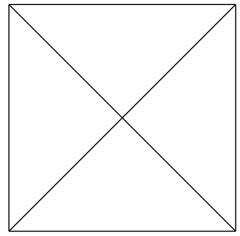

Figure 2: $BC(1,1)$: a 1-reticulated square with four cells.

example, we might join every vertex or node to all the vertices or nodes on all the other sides. Points where chords intersect are also nodes of the graph G. The graph thus formed has *nodes* (referring to the vertices and edge-nodes of the polygon and also any interior intersection points), *edges* (which are line segments between pairs of nodes), and *cells* (the connected regions defined by the edges). In graph theory the cells are sometimes called *faces* or *chambers*, but we will not use those terms. Our graphs are also *maps* in the sense of Tutte [24, 25], but we will refer to them simply as planar graphs.

For a connected planar graph, Euler's formula states that the numbers of nodes, edges, and cells are related by

$$|\text{nodes}| - |\text{edges}| + |\text{cells}| = 1. \tag{1.1}$$

Examples: Figure 2 shows the graph $BC(1,1)$ (defined in §2), which has 5 nodes, 8 edges, and 4 cells. The polygon is a square and there are two chords which meet at the central node. Figure 3 shows the graph $BC(2,2)$, constructed from a square in which each side has been 2-reticulated. There are 14 chords. The graph has 37 nodes, 92 edges, and 56 cells.[4] Figure 4 shows a colored version of the graph. The principles used to color these graphs are discussed in Section 10. For any undefined terms from graph theory see [4, 9].

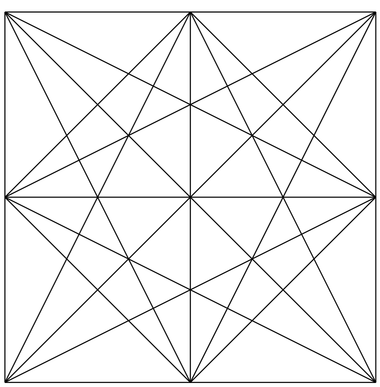

Figure 3: $BC(2,2)$: a 2-reticulated square with 56 cells.

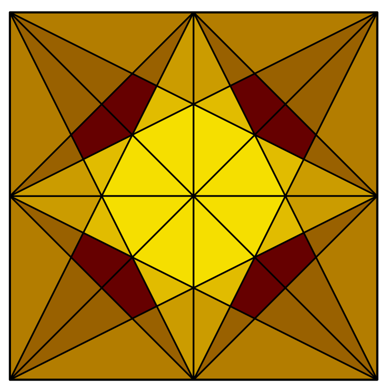

Figure 4: The same $BC(2,2)$ drawn with colored cells. See §10.2 for coloring scheme.

2 $BC(1,n)$: $1 \times n$ rectangular windows

The graph $BC(1,n)$ $(n \geq 1)$ is constructed by taking a $1 \times n$ rectangle, inserting $n-1$ equally spaced nodes along the top and bottom sides, and then joining every pair of vertices or nodes by chords. Figures 2, 5, and

[4] $BC(3,3)$ is shown in Fig. 14 in §6 and has 340 cells. There is no known formula for the number of cells in $BC(n,n)$, even though we have 52 terms. The sequence begins $4, 56, 340, 1120, 3264, \ldots$ (A255011).

6 show $BC(1, n)$ for $n = 1, 2$, and 3.

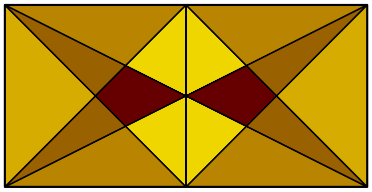

Figure 5: $BC(1, 2)$.

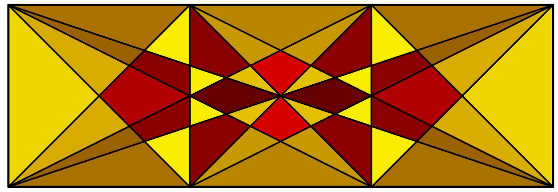

Figure 6: $BC(1, 3)$.

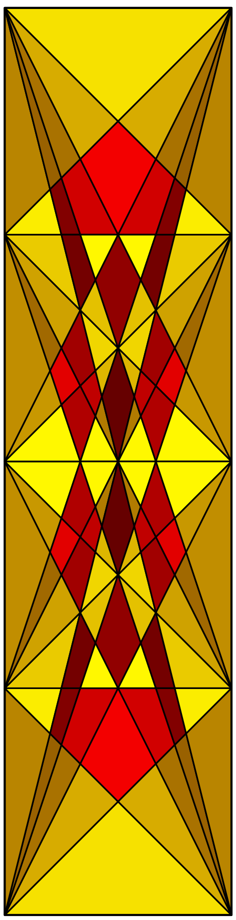

Figure 7: $BC(4, 1)$ colored using to the red and yellow palettes (see §10.2).

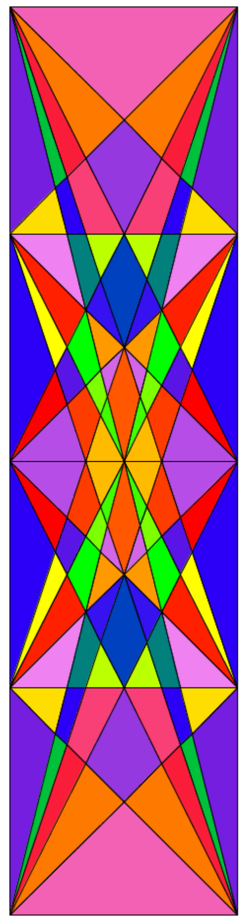

Figure 8: A version of $BC(4, 1)$ colored by our 'random coloring' algorithm (see §10.3).

Of course we could equally well have started with a vertical rectangle of size $n \times 1$, in which case the graph would be denoted by $BC(n, 1)$. Since this work was partly inspired by the windows of Gothic cathedrals, we admit to a slight preference for $BC(n, 1)$ over $BC(1, n)$, although as graphs they are isomorphic. Figs. 7 and 8 show our stained glass window $BC(4, 1)$ using two different coloring schemes.

We will continue to discuss $BC(1, n)$, but the reader should remember that the results apply equally well to $BC(n, 1)$.

Another way to construct $BC(1,n)$ is to start with the complete bipartite graph $K_{n+1,n+1}$ formed by taking $n+1$ equally spaced points in each of two horizontal rows, joining every upper point to every lower point, and then adding the line segments through the two rows of points. Thus $BC(1,2)$ in Fig. 5 is the well-known nonplanar "utilities" graph $K_{3,3}$ if the two horizontal lines and the colors are ignored.

The graphs $BC(1,n)$ are one of the few families where there are explicit formulas for the numbers of nodes ($\mathcal{N}(1,n)$), edges ($\mathcal{E}(1,n)$), and cells ($\mathcal{C}(1,n)$). The initial values of these quantities are shown in Table 1, along with the A-numbers in [14]) of the corresponding sequences.

Table 1: Numbers of nodes, edges, cells in $BC(1,n)$.

$n:$	1	2	3	4	5	6	7	8	9	10	$\cdots$	[14]
$\mathcal{N}(1,n):$	5	13	35	75	159	275	477	755	1163	1659	$\cdots$	A331755
$\mathcal{E}(1,n):$	8	28	80	178	372	654	1124	1782	2724	3914	$\cdots$	A331757
$\mathcal{C}(1,n):$	4	16	46	104	214	380	648	1028	1562	2256	$\cdots$	A306302

Since by Euler's formula (1.1), $\mathcal{E}(1,n) = \mathcal{N}(1,n) + \mathcal{C}(1,n) - 1$, there is no need to tabulate $\mathcal{E}(1,n)$, and in future we shall omit those numbers.

The following theorem is due to Legendre (2009) [10] and Griffiths (2010) [7], who discuss the problem from the point of view of $K_{n+1,n+1}$. First we introduce an expression that will frequently appear in these formulas. For $m,n,q \ge 1$, let

$$V(m,n,q) \;=\; \sum_{a=1..m} \;\; \sum_{\substack{b=1..n \\ \gcd\{a,b\}=q}} (m+1-a)(n+1-b)\,. \tag{2.1}$$

Theorem 2.1. *(Legendre [10, Prop. 6], Griffiths [7, Th. 3].) For $n \ge 1$, the number of nodes in $BC(1,n)$, $\mathcal{N}(1,n)$ (A331755) is given by*

$$\mathcal{N}(1,n) \;=\; 2(n+1) + V(n,n,1) - V(n,n,2)\,, \tag{2.2}$$

and the number of cells, $\mathcal{C}(1,n)$ (A306302) is

$$\mathcal{C}(1,n) \;=\; n^2 + 2n + V(n,n,1)\,. \tag{2.3}$$

Remarks: (i) A key step in the proof of (2.2) (see [10]) is finding a condition for three chords to meet at a point. (ii) The starting point for the proof of (2.3) (see [7]) is the observation that in the graph $BC(1,n)$ there are no interior edges that are parallel to the two long sides of the rectangle. This means that every cell has a unique node that is closest to the upper side of the rectangle. (iii) The term $2(n+1)$ on the right-hand side of (2.2) is the number of nodes on the boundary of the rectangle. The difference between the other two terms is therefore the number of interior nodes in $BC(1,n)$ (A159065):

$$1, 7, 27, 65, 147, 261, 461, 737, 1143, \cdots. \tag{2.4}$$

In 2019 Max Alekseyev added a comment to A306302 pointing out that the results in Theorem 2.1 are essentially the same as the results he and his coauthors had obtained in [2] (2015) for the isosceles triangle graphs $IT(n)$.

3 The isosceles triangle graph $IT(n)$.

The definition of the *isosceles triangle graph* $IT(n)$, $n \ge 1$, starts with an isosceles right triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$. On the vertical side of the triangle we place n nodes at the points

$$\left(0,\frac{1}{2}\right), \left(0,\frac{1}{3}\right), \left(0,\frac{1}{4}\right), \ldots, \left(0,\frac{1}{n+1}\right),$$

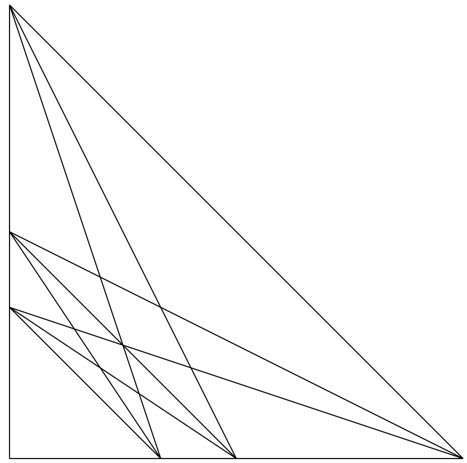

Figure 9: The isosceles triangle graph $IT(2)$. There are 14 nodes (7 on boundary, 7 in interior), 17 cells (15 triangles, 2 quadrilaterals), and 30 edges.

and similarly on the horizontal side we place n nodes at the points

$$\left(\frac{1}{2},0\right),\left(\frac{1}{3},0\right),\left(\frac{1}{4},0\right),\ldots,\left(\frac{1}{n+1},0\right).$$

There are no internal nodes on the hypotenuse.[5] We then draw chords between every pair of the $2n+3$ points on the boundary of the triangle. Figs. 9, 10, 11 show $IT(2)$, $IT(3)$ and $IT(4)$. The latter two graphs have been colored using the red and yellow palettes (§10.2).

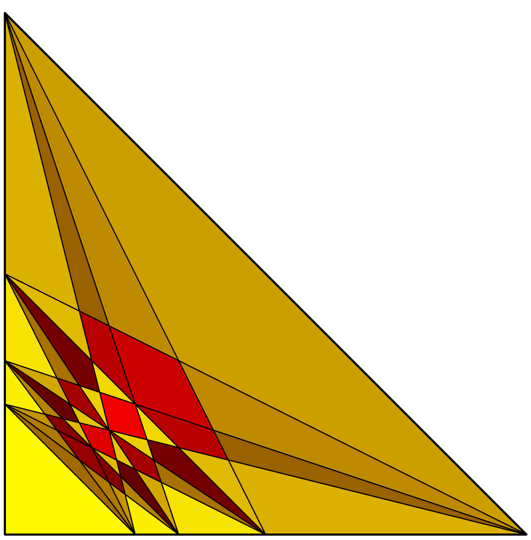

Figure 10: $IT(3)$ (33 triangles, 14 quadrilaterals)

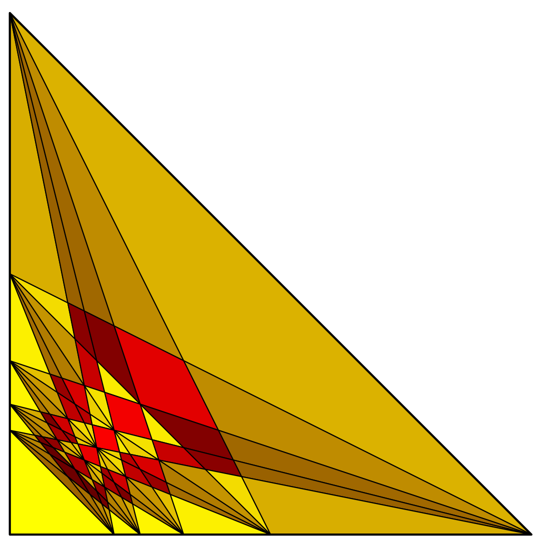

Figure 11: $IT(4)$ (71 triangles, 34 quadrilaterals)

Alekseyev pointed out that if we take the boundary points of $BC(1,n)$ to be the points $(i,0)$ and $(i,1)$ for

[5] In Part 2 of this paper [3] we will discuss graphs formed by inserting n equally-spaced nodes on all three sides of an equilateral triangle.

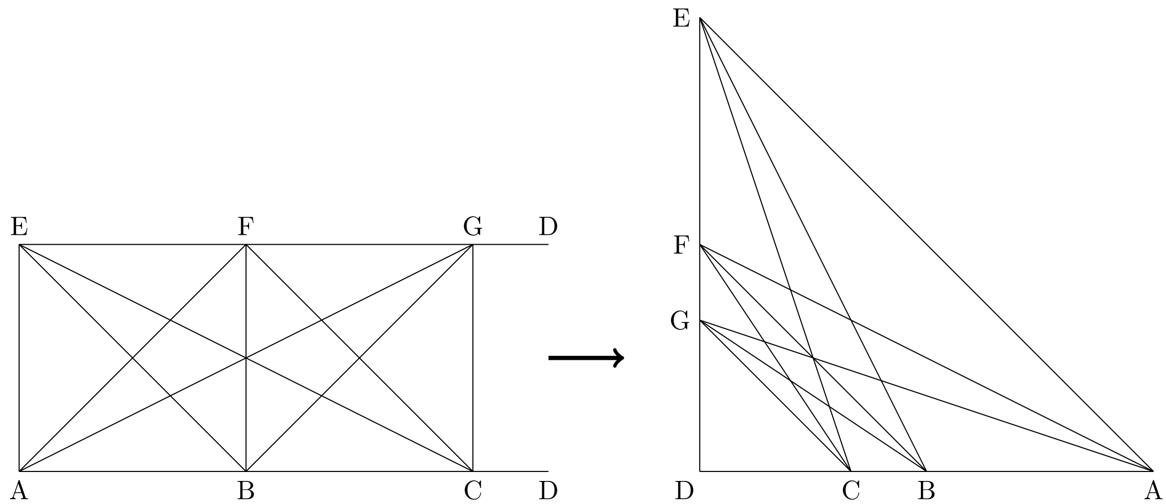

Figure 12: Illustrating the map (3.1) from $BC(1,2)$ to $IT(2)$.

$i = 0, \ldots, n$, then the map

$$(x, y) \longmapsto \left(\frac{1-y}{x+1}, \frac{y}{x+1}\right), \tag{3.1}$$

maps $BC(1,n)$ onto $IT(n)$ minus the node and cell at the origin. Figure 12 illustrates this in the case $n = 2$. The six boundary nodes A, B, C, E, F, G of $BC(1,2)$ are mapped to six of the seven boundary nodes of $IT(2)$. The point D, the point at infinity on the positive x axis (not part of $BC(1,2)$), is mapped to the origin in $IT(2)$. The region D, C, G, D to the right of $BC(1,2)$ is mapped to the triangular cell D, G, C, D at the origin in $IT(2)$.

A similar thing happens in the general case: $IT(n)$ always has one more node than $BC(1,n)$, two more edges, and one more cell. When these adjustments are made to the formulas in Theorem 2.1, we obtain the formulas in Theorem 13 of [2]. The counts for nodes, edges, and cells in $IT(n)$ are given in A332632, A332360, and A332358.

However, Alekseyev (personal communication) also pointed out that Theorem 13 of [2] mentions an additional property of $IT(n)$—and hence of $BC(1,n)$—that seems to have been overlooked in [10] and [7]:

Theorem 3.1. *(Alekseyev et al. [2]): The cells in $IT(n)$ and hence $BC(1,n)$ are either triangles or quadrilaterals.*

That is, no cell in $BC(1,n)$ has five or more edges. The proof in [2] depends on a theorem about teaching sets for threshold function [19, 26]. No other proof seems to be known. We therefore state:

Open Problem 3.2. *Find a purely geometrical proof of Theorem 3.1.*

4 The cells in $BC(1,n)$

From Theorems 2.1 and 3.1, we can determine the numbers of triangular and quadrilateral cells in $BC(1,n)$ (sequences A324042 and A324043).

Theorem 4.1. *The $\mathcal{C}(1,n)$ cells in $BC(1,n)$ are made up of*

$$T(n) = 2V(n,n,2) + 2n(n+1) \tag{4.1}$$

triangles and

$$Q(n) = V(n,n,1) - 2V(n,n,2) - n^2 \tag{4.2}$$

quadrilaterals.

Proof. The sum $3T(n)+4Q(n)$ double-counts the edges in $BC(1,n)$ except that the $2n+2$ boundary edges are counted only once. Therefore

$$3T(n) + 4Q(n) + (2n+2) = 2\mathcal{E}(1,n) = 2(\mathcal{N}(1,n) + \mathcal{C}(1,n) - 1)\,, \tag{4.3}$$

and of course by Theorem 3.1, $T(n)+Q(n) = \mathcal{C}(1,n)$. The proof is completed by solving these two equations for $T(n)$ and $Q(n)$ and using (2.2), (2.3). ☐

Figures 2, 5, 6, and 7 show the triangles and quadrilaterals for $n = 1, \ldots, 4$.

One way to attack Open Problem 3.2 is to try to understand the distribution of cells in each of the n squares of $BC(1,n)$. Let $t_{n,k}$, $q_{n,k}$, and $c_{n,k}$ denote the numbers of triangles, quadrilaterals, and cells in the k-th square of $BC(1,n)$ for $1 \le k \le n$ (so $t_{n,k} + q_{n,k} = c_{n,k}$ and $\sum_k c_{n,k} = \mathcal{C}(1,n)$). From Fig. 5, for example, we see that $t_{1,1} = t_{1,2} = 7$, $q_{1,1} = q_{1,2} = 1$, and $c_{1,1} = c_{1,2} = 8$.

The two end squares of $BC(1,n)$ are easily understood, and for future reference we state the result as:

Theorem 4.2. *For $n \ge 2$, the two end squares of $BC(1,n)$ both contain $2n+3$ triangles and $2n-3$ quadrilaterals.*

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Table 2: Number $t_{n,k}$ of triangles in k-th square in $BC(1,n)$ (A333286).

$n \backslash k$	1	2	3	4	5	6	7	8	9	10
1	4									
2	7	7								
3	9	14	9							
4	11	24	24	11						
5	13	30	38	30	13					
6	15	38	60	60	38	15				
7	17	44	76	86	76	44	17			
8	19	52	92	120	120	92	52	19		
9	21	58	106	146	158	146	106	58	21	
10	23	66	126	178	216	216	178	126	66	23

Tables 2, 3, and 4 show the values of $t_{n,k}$, $q_{n,k}$, and $c_{n,k}$ for $n \le 10$. More extensive tables, for $n \le 80$, are given in entries A333286, A333287, A333288. However, even with 80 rows of data, we have been unable to find formulas for these numbers.

There is certainly a lot of structure in these tables. Using the Salvy-Zimmermann *gfun* Maple program [18], we attempted to find generating functions for the columns of these tables. On the basis of admittedly little evidence, we make the following conjecture.

Conjecture 4.3. *In all three of Tables 2, 3, and 4, the k-th column for $k \ge 3$ has a rational generating function which can be written with denominator $(1-x^{k-2})(1-x^{k-1})(1-x^k)$.*

Table 3: Number $q_{n,k}$ of quadrilaterals in k-th square in $BC(1,n)$ (A333287).

$n \backslash k$	1	2	3	4	5	6	7	8	9	10
1	0									
2	1	1								
3	3	8	3							
4	5	12	12	5						
5	7	22	32	22	7					
6	9	28	40	40	28	9				
7	11	38	58	74	58	38	11			
8	13	46	74	98	98	74	46	13		
9	15	58	92	130	152	130	92	58	15	
10	17	68	104	150	180	180	150	104	68	17

Table 4: Total number $c_{n,k}$ of cells in k-th square in $BC(1,n)$ (A333288).

$n \backslash k$	1	2	3	4	5	6	7	8	9	10
1	4									
2	8	8								
3	12	22	12							
4	16	36	36	16						
5	20	52	70	52	20					
6	24	66	100	100	66	24				
7	28	82	134	160	134	82	28			
8	32	98	166	218	218	166	98	32		
9	36	116	198	276	310	276	198	116	36	
10	40	134	230	328	396	396	328	230	134	40

For example, column 3 of Table 2, the sequence $\{t_{n,3}\}$, appears to have generating function

$$x^3 \ \frac{9 + 15x + 5x^2 - 2x^3 - 13x^4 - 11x^5 - 9x^6 + 2x^7 + 8x^8 - 4x^{10} + 4x^{12}}{(1-x)(1-x^2)(1-x^3)} \,. \tag{4.4}$$

It would be nice to know more about these quantities.

5 The nodes in $BC(1,n)$

Besides looking at the cells of $BC(1,n)$, it is also interesting to study the nodes. For $n \geq 2$, $BC(1,n)$ has four boundary nodes of degree $n+1$ and $2n-2$ boundary nodes of degree $n+2$. An interior node formed when c chords (say) cross has degree $2c$. Let $v_{n,c}$ denote the number of interior nodes of degree $2c$, for $2 \leq c \leq n+1$. Table 5 shows the values of $v_{n,c}$ for $n \leq 10$. A more extensive table, for $n \leq 100$, is given in A333275.

Table 5: Number $v_{n,c}$ of interior nodes in $BC(1,n)$ where c chords cross (A333275).

$n \backslash c$	2	3	4	5	6	7	8	9	10	11
1	1									
2	6	1								
3	24	2	1							
4	54	8	2	1						
5	124	18	2	2	1					
6	214	32	10	2	2	1				
7	382	50	22	2	2	2	1			
8	598	102	18	12	2	2	2	1		
9	950	126	32	26	2	2	2	2	1	
10	1334	198	62	20	14	2	2	2	2	1

Theorem 5.1. *For $n \geq 2$, the numbers $v_{n,c}$ satisfy:*

$$\sum_{c=2}^{n+1} v_{n,c} \; + \; 2n+2 \; = \; \mathcal{N}(1,n)\,, \tag{5.1}$$

$$\sum_{c=2}^{n+1} c\, v_{n,c} \; + \; n^2+4n+1 \; = \; \mathcal{E}(1,n)\,, \tag{5.2}$$

$$\sum_{c=2}^{n+1} \binom{c}{2} v_{n,c} \; = \; \binom{n+1}{2}^2 . \tag{5.3}$$

Proof. The first equation simply gives the total number of nodes in $BC(1,n)$. For (5.2) we count pairs (α,β), where α is a cell and β is a node, in two ways, obtaining

$$3T(n) + 4Q(n) \; = \; 4(n+1) + (2n-2)(n+2) + \sum_c 2c\, v_{n,c}\,,$$

and use (4.3). To establish (5.3), we start with the observation that if all the $2n+2$ boundary points of $BC(1,n)$ are perturbed by small random amounts, there will be no triple or higher-order intersection points, all the internal nodes will be simple, and there will be $\binom{n+1}{2}^2$ of them (since any pair of nodes on the upper side of the rectangle and any pair of nodes on the lower side will determine a unique intersection point). As the boundary points are returned to their true positions, the internal nodes coalesce. If there is an interior point where c chords intersect, the $\binom{c}{2}$ interior nodes there coalesce into one, and we lose $\binom{c}{2}-1$ intersections. We are left with the $\mathcal{N}(1,n)-(2n+2)$ interior intersection points. Thus

$$\sum_{c=2}^{n+1} \left(\binom{c}{2} - 1\right) v_{n,c} \; + \; \mathcal{N}(1,c) - (2n+2) \; = \; \binom{n+1}{2}^2,$$

which simplifies to give (5.3). □

However, we do not even have a formula for the number of simple interior intersection points in $BC(1,n)$ (the first column of Table 5, the sequence $\{v_{n,2}\}$, A334701), although we have computed 500 terms. The first 100 terms are shown in Table 6. We feel that a formula should exist!

Open Problem 5.2. *Find a formula for the number of simple interior intersection points in $BC(1,n)$ (see Table 6 for 100 terms, or A334701 for 500 terms).*

Table 6: The first 100 terms of the number of simple interior intersection points in $BC(1, n)$.

Terms 1–25	26–50	51–75	76–100
1	49246	679040	3264422
6	57006	732266	3438642
24	65334	790360	3616430
54	75098	849998	3805016
124	85414	914084	3998394
214	97384	980498	4202540
382	110138	1052426	4408406
598	124726	1125218	4626162
950	139642	1203980	4850198
1334	156286	1285902	5085098
1912	174018	1374300	5321854
2622	194106	1463714	5571470
3624	214570	1559064	5826806
4690	237534	1657422	6095870
6096	261666	1762004	6369534
7686	288686	1869106	6655902
9764	316770	1983922	6948566
12010	348048	2102162	7256076
14866	380798	2228512	7565826
18026	416524	2356822	7889032
21904	452794	2493834	8220566
25918	492830	2635310	8568428
30818	534962	2786090	8919298
36246	580964	2938326	9285288
42654	627822	3099230	9658638

6 $BC(m, n)$: $m \times n$ rectangular windows

The graph $BC(1, n)$ $(n \geq 1)$ is based on a $1 \times n$ rectangle. In this section we consider what happens if we start more generally from an (m, n)-reticulated rectangle (where $m \geq 1$, $n \geq 1$): this is a rectangle of size $m \times n$ in which both vertical edges are divided into m equal parts, and both horizontal edges into n equal parts. There are $m - 1$ nodes on each vertical edge and $n - 1$ nodes on each horizontal edge, for a total of $4 + 2(m - 1) + 2(n - 1) = 2(m + n)$ boundary nodes.

We will discuss three families of graphs based on these rectangles, denoted by $BC(m, n)$, $AC(m, n)$, and $LC(m, n)$. The graph $BC(m, n)$ is formed by joining every pair of boundary nodes by a line segment and placing a node at each point where two or more line segments intersect. Figs. 3 and 4 show $BC(2, 2)$, and Fig. 14 shows $BC(3, 3)$. ("BC" stands for "boundary chords".)

Alternatively, we could have constructed $BC(m, n)$ by starting with an $m \times n$ grid of equal squares, and then joining each pair of boundary grid points by a line segment. However, if we include the interior grid points, there are there are $(m + 1)(n + 1)$ grid points in all, and if we join *each* pair of grid points by a line segment, we obtain the graph $AC(m, n)$. ("AC" stands for "all chords".) These graphs are discussed by Huntington T. Hall [8], Marc E. Pfetsch and Günter M. Ziegler [15], and Hugo Pfoertner (entry A288187 in [14]). We shall say more about $AC(m, n)$ in §8.

A third family of graphs, $LC(m, n)$, arises if we extend each line segment in $AC(m, n)$ until it reaches the boundary of the grid. ("LC" stands for "long chords".) These graphs are discussed by Seppo Mustonen

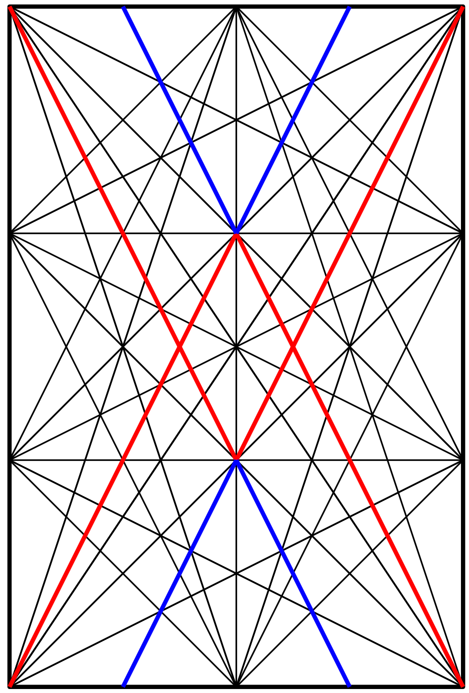


Figure 13: Comparison of the graphs $BC(3,2)$ (black lines), $AC(3,2)$ (add the red lines), and $LC(3,2)$ (also add the blue lines).

[11, 12, 13]. We say more about $LC(m,n)$ in §9.

Figure 13 shows the differences between the three definitions in the case of a $(3,2)$ reticulated rectangle, the first time the definitions differ. The black lines form the graph $BC(3,2)$. The four red lines are the additional line segments that appear when we construct $AC(3,2)$. They start at an interior grid point and so are not present in $BC(3,2)$. The four blue lines extend the red chords until they reach the boundary of the grid, and form $AC(3,2)$.

The numbers of nodes $\mathcal{N}(m,n)$ and cells $\mathcal{C}(m,n)$ in $BC(m,n)$ are shown for $m,n \le 37$ in A331453 and A331452, respectively, and the initial terms are shown in Table 7.

Table 7: Numbers of nodes $\mathcal{N}(m,n)$ and cells $\mathcal{C}(m,n)$ in $BC(m,n)$ for $1 \le m,n \le 7$.

$m \backslash n$	1	2	3	4	5	6	7
1	5, 4	13, 16	35, 46	75, 104	159, 214	275, 380	477, 648
2	13, 16	37, 56	99, 142	213, 296	401, 544	657, 892	1085, 1436
3	35, 46	99, 142	257, 340	421, 608	881, 1124	1305, 1714	2131, 2678
4	75, 104	213, 296	421, 608	817, 1120	1489, 1916	2143, 2820	3431, 4304
5	159, 214	401, 544	881, 1124	1489, 1916	2757, 3264	3555, 4510	5821, 6888
6	275, 380	657, 892	1305, 1714	2143, 2820	3555, 4510	4825, 6264	7663, 9360
7	477, 648	1085, 1436	2131, 2678	3431, 4304	5821, 6888	7663, 9360	12293, 13968

Regrettably, except when m or n is 1, we have been unable to find formulas for any of these quantities. The diagonal case, when $m = n$, is the most interesting (because the most symmetrical), but is also probably

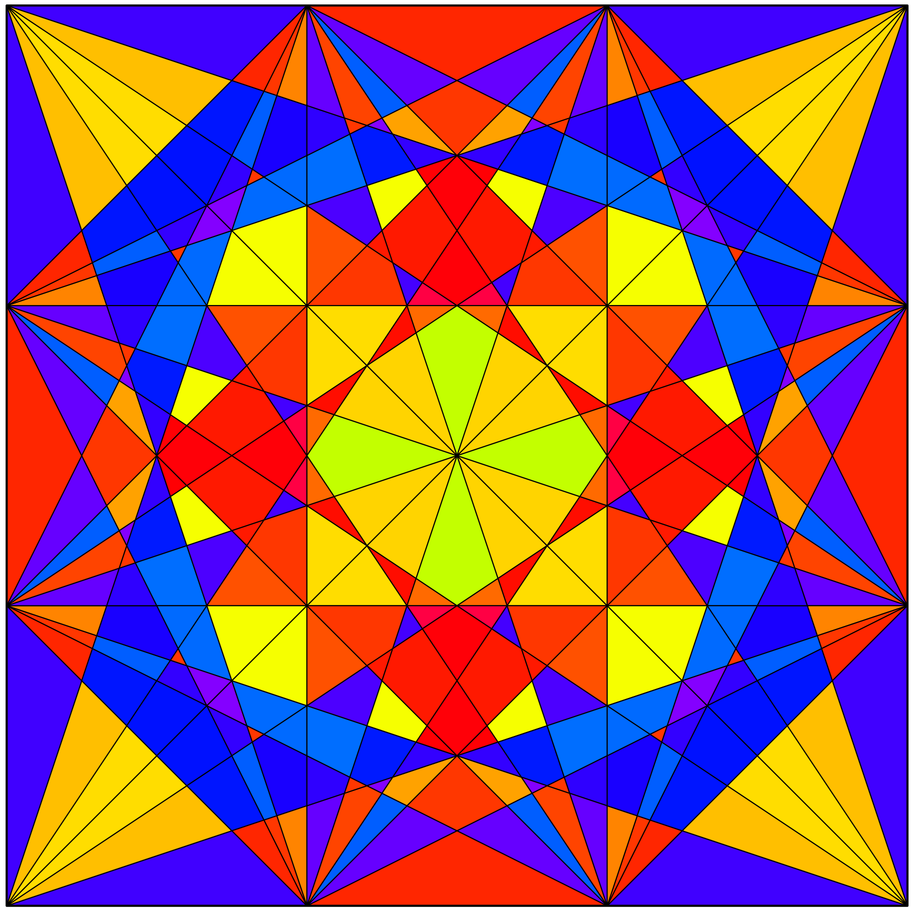


Figure 14: The graph $BC(3,3)$. There are 257 nodes and 340 cells.

the hardest to solve. In accordance with our philosophy of "if you can't solve it, make art", Fig. 14 shows our stained glass window $BC(3,3)$, and entry A331452 has a large number of larger and even more striking examples which space restrictions do not permit us to show here.

Out of all these unsolved problems, the case $m = 2$ (or $n = 2$) would seem to be the most amenable to analysis, perhaps by extending the work of Legendre [10] and Griffiths [7]. For instance, what are the conditions for three chords in $BC(2,n)$ to intersect at a common point? We emphasize this by stating:

Open Problem 6.1. *Find formulas for the numbers of nodes ($\mathcal{N}(2,n)$, A331763) and cells ($\mathcal{C}(2,n)$, A331766) in $BC(2,n)$.*

The first 10 terms are given in Table 8, and 100 terms are given in the entries for these two sequences in [14].

Table 8: Numbers of nodes and cells in $BC(2,n)$.

n:	1	2	3	4	5	6	7	8	9	10	⋯	[14]
$\mathcal{N}(2,n)$:	13	37	99	213	401	657	1085	1619	2327	3257	⋯	A331763
$\mathcal{C}(2,n)$:	16	56	142	296	544	892	1436	2136	3066	4272	⋯	A331766

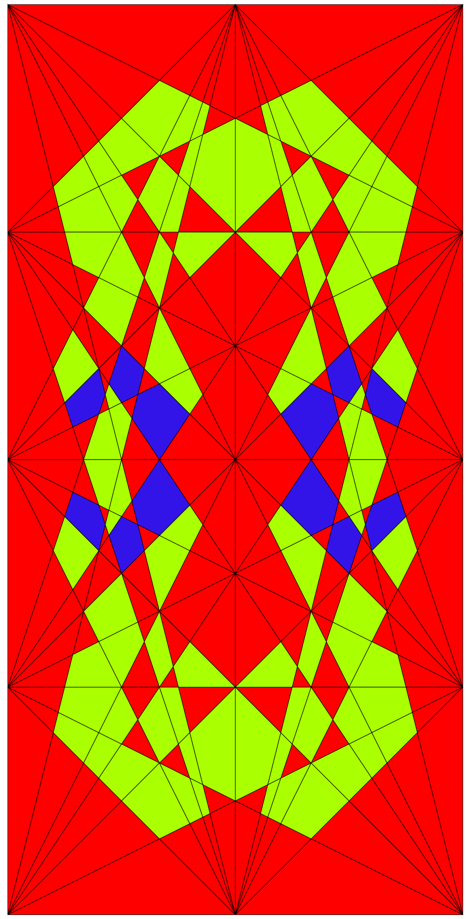


Figure 15: $BC(4,2)$ with cells color-coded to distinguish triangles (red), quadrilaterals (yellow), and pentagons (blue).

In $BC(1,n)$ the cells are always triangles or quadrilaterals (Theorem 3.1). It appears that a similar phenomenon holds for $BC(2,n)$. The data strongly suggests the following conjecture.

Conjecture 6.2. *The cells in $BC(2,n)$ have at most eight sides, and for $n \ge 19$, at most six sides.*

We have verified the conjecture for $n \le 106$. Row n of Table 9 gives the number of cells in $BC(2,n)$ with k sides, for $k \ge 3$ and $n \le 20$. For rows $n = 1, 2$, and 3 of this table see Figs. 5, 2, and 13 (black lines only). For row 4 see Figure 15, where one can see that $BC(4,2)$ has 192 triangular cells (red), 92 quadrilaterals (yellow), and 12 pentagons (blue). Entry A335701 gives the first 106 rows of this table, and has many further illustrations. The row sums in Table 9 are the numbers $\mathcal{C}(2,n)$ given in column 2 of Table 7 and A331766.

More generally we may ask: For $BC(m,n)$, m fixed, is there an upper bound on the number of sides of a cell as n varies?

We are at least able to analyze the corner squares of $BC(2,n)$.

Theorem 6.3. *For $n = 2$ the four corner squares of $BC(2,n)$ (and $BC(n,2)$) each contain 12 triangles and 4 quadrilaterals, while for $n = 3$ they contain 15 triangles, 6 quadrilaterals, and (exceptionally) one pentagon. For $n \ge 4$, the corner squares each contain $7n+1$ cells, consisting of $2n+9$ triangles and $5n-8$ quadrilaterals.*

Table 9: Row n gives the number of cells in $BC(2, n)$ with k sides, for $k \geq 3$. It appears that for $n \geq 19$, no cell has more than six sides (see A335701).

$n\backslash k$	3	4	5	6	7	8
1	14	2				
2	48	8				
3	102	36	4			
4	192	92	12			
5	326	194	24			
6	524	336	28	4		
7	802	554	80			
8	1192	812	128	4		
9	1634	1314	112	0	4	2
10	2296	1756	200	20		
11	3074	2508	236	22		
12	4052	3252	356	28		
13	5246	4348	472	28		
14	6740	5464	652	28		
15	8398	7054	656	74		
16	10440	8760	940	52		
17	12770	11050	1040	58		
18	15512	13324	1300	60	4	
19	18782	16162	1600	70		
20	22384	19256	1948	104		

Proof. We consider the top left corner square of $BC(n, 2)$, assuming $n \geq 4$. The key to the proof is to dissect this square into regions, in each of which the cell structure is apparent, and such that the boundaries of the regions do not cross any cell boundaries. This is done as indicated in Fig. 16. There are six regions, labeled a through f.

We assume the coordinates are chosen so that nodes A, B, C, D have coordinates $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, respectively. The four vertices of the rectangle defining $BC(n, 2)$ have coordinates $(0, 0)$, $(2, 0)$, $(2, n)$, and $(0, n)$.

The chord from A to the grid point $(1, n)$ cuts the line DC midway between D and F, and the $n - 1$ chords from A to grid points $(2, n)$, $(2, n - 1), \ldots,$ $(2, 2)$ cut DC between F and C. The final chord from A to $(2, 1)$ cuts BC at E. The top left triangular region f is therefore divided into $n + 2$ triangular cells.

There is a chord from B to D, a chord from B to the grid point $(0, 2)$ which cuts DC at F, and $n - 2$ further chords from B to the grid points $(0, 3), \ldots, (0, n)$, which cut DC to the right of F.

There is one further chord that cuts this corner square, the chord fron D to E

The reader will now have no difficulty in verifying that the cells in regions a, b, c, d, e, f are as shown in Table 10.

□

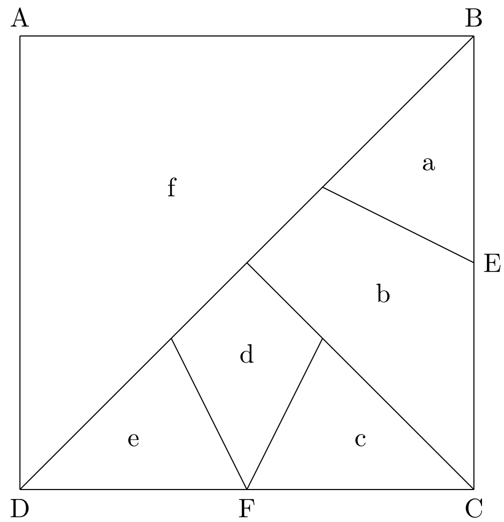


Figure 16: Dissection of corner square of $BC(n,2)$, $n \geq 4$, used in proof of Theorem 6.3.

Table 10: Numbers of triangles and quadrilaterals in the regions shown in Fig. 16.

Region	triangles	quadrilaterals
a	n	0
b	2	$2n-3$
c	2	$n-2$
d	1	3
e	2	$2n-6$
f	$n+2$	0
Total	$2n+9$	$5n-8$

7 $BC(m,n)$ in general position.

We can obtain reasonably good upper bounds on $\mathcal{N}(m,n)$ and $\mathcal{C}(m,n)$ by analyzing what would happen if all the intersection points in $BC(m,n)$ were simple intersections—that is, if there was no interior point where three or more chords met.

We use $BC_{GP}(m,n)$ to denote a graph obtained by perturbing the boundary nodes of $BC(m,n)$ (excluding the four vertices) by small random sideways displacements along the boundaries. That is, if a boundary node was a fraction $\frac{i}{j}$ say of the way along an edge, we move it to a point $\frac{i}{j}+\epsilon$ of the way along the edge, where ϵ is a small random real number. If the ϵ's are chosen independently, the new graph will be in "general position", and there will be no multiple intersection points in the interior.

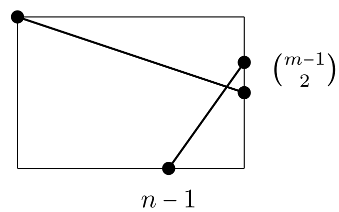
To illustrate the perturbing process, in Fig. 17 below one can see (ignoring for now the supporting strut on the left) a perturbed version of $BC(1,2)$ obtained by slightly displacing just one node (labeled 4) so as to avoid the triple intersection point at the center (see Fig. 5).

Let $\mathcal{N}_{GP}(m,n)$ and $\mathcal{C}_{GP}(m,n)$ denote the numbers of nodes and cells, respectively, in the perturbed graph. The perturbations increase the numbers of nodes and cells, so $\mathcal{N}_{GP}(m,n) \geq \mathcal{N}(m,n)$ and $\mathcal{C}_{GP}(m,n) \geq \mathcal{C}(m,n)$, and $\mathcal{N}_{GP}(m,n) \to \mathcal{N}(m,n)$ and $\mathcal{C}_{GP}(m,n) \to \mathcal{C}(m,n)$ as the displacements are reduced to zero.

Theorem 7.1. *For $m, n \ge 1$, the number of interior nodes in $BC_{GP}(m,n)$ is*

$$\frac{1}{4}\{(m+n)(m+n-1)^2(m+n-4) \;+\; 2mn(2m+n-1)(m+2n-1)\}\,. \tag{7.1}$$

Proof. We start with the observation that any four boundary nodes of the rectangle, no three of which are on an edge, determine a unique intersection point in the interior of the rectangle. There are several ways to choose these four points. They might be the four vertices of the rectangle, which can be done in just one way. They might consist of three vertices and a single node on one of the other two sides, which can be done in $4(m_1+n_1)$ ways, where $m_1 = m-1$ and $n_1 = n-1$ are the numbers of ways of choosing a single non-vertex point on a side. A more typical example consists of one vertex, and one, resp. two, points on the two opposite sides, as shown in the following drawing. This can be done in $4(m_1n_2+m_2n_1)$ ways, where $m_2 = (m-1)(m-2)/2$, $n_2 = (n-1)(n-2)/2$ are the numbers of ways of choosing two non-vertex nodes from the sides.



There are in all seventeen different configurations for choosing four points, and when the seventeen counts are added up the result is the expression given in (7.1). We omit the details. ☐

Remarks. (i) Since there are $2(m+n)$ boundary nodes, the total number of nodes in $BC_{GP}(m,n)$ is

$$\mathcal{N}_{GP}(m,n) \;=\; \frac{1}{4}\{(m+n)(m+n-1)^2(m+n-4) \;+\; 2mn(2m+n-1)(m+2n-1)\} + 2(m+n)\,. \tag{7.2}$$

This is our upper bound for $\mathcal{N}(m,n)$.

(ii) Another way to interpret $\mathcal{N}_{GP}(m,n)$ is that this is the number of nodes in $BC(m,n)$ counted with multiplicity (meaning that if there is an interior node where c chords meet, it contributes $c-1$ to the total).

(iii) When $m = n$, (7.2) simplifies to

$$\frac{n}{2}\,(17n^3 - 30n^2 + 19n + 4)\,, \tag{7.3}$$

which is our upper bound for $\mathcal{N}(n,n)$. For $n = 52$, $\mathcal{N}(n,n) = 52484633$ (from A331449), while (7.3) gives 60065408, too large by a factor of 1.14, which is not too bad. The moral seems to be that most internal nodes are simple.

(iv) When $m = 1$, (7.2) becomes $n^2(n+1)^2/4$, which agrees with the number mentioned in the proof of Theorem 5.1.

(v) For large m and n, the expression (7.2) is dominated by the degree 4 terms, which are

$$\frac{1}{4}\left(m^4 + n^4 + 8mn(m^2+n^2) + 16m^2n^2\right)\,. \tag{7.4}$$

Setting $m = n$, we get $\mathcal{N}_{GP}(n,n) \sim 17n^4/2$ as $n \to \infty$. We can confirm this by looking at the number of ways to choose four nodes out of the $4n$ boundary nodes so that no three are on a side. This is (essentially)

$$\binom{4n}{n} \;-\; 4\binom{n}{4} \;-\; 12\,n\binom{n}{3} \;\sim\; \frac{17}{2}n^4\,.$$

(vi) From (v), we have $\mathcal{N}(n,n) = O(n^4)$. In fact, we conjecture that $\mathcal{N}(n,n) \sim \mathcal{N}_{BC}(n,n) \sim 17\,n^4/2$. But to establish this we would need better information about the number of interior nodes in $BC(n,n)$ with a given multiplicity.

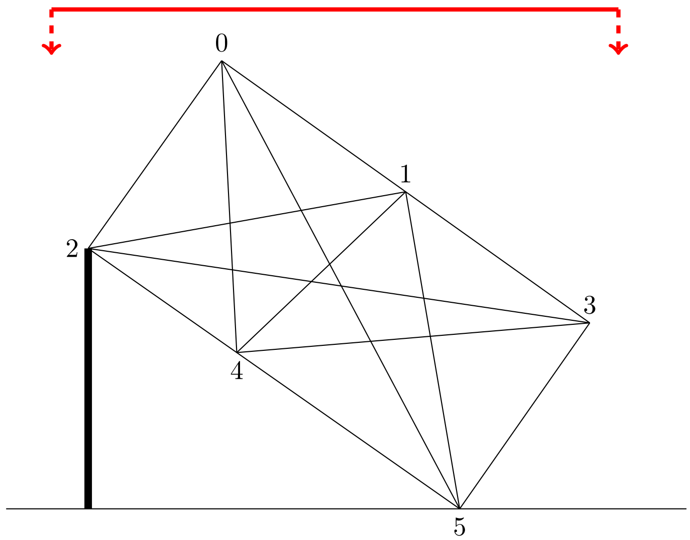


Figure 17: $BC(1,2)$ in general position: node 4 has been displaced slightly so as to avoid the triple intersection point at the center. The strut on the left tilts the figure so that the ordinates of the boundary nodes are in the same order as the labels. The red line is the "counting line", which descends across the picture in order to count the cells.

Now that we know the number of nodes, we can also find the number $\mathcal{C}_{GP}(m,n)$ of cells in $BC_{GP}(m,n)$. For this we use a method described by Freeman [6]. The following is a slight modification of his procedure. $BC_{GP}(m,n)$ has $2(m+n)$ boundary nodes. We label the top left corner vertex 0, and the bottom right corner vertex $2(m+n)-1$. The nodes along the top edge we label 0, 1, 3, 5 . . .,$2n-1$, continuing along along the right-hand edge with $2n+1$, $2n+3$ . . ., $2(m+n)-1$. Along the left-hand edge we place the labels 0, 2, 4, . . ., $2m$, continuing along the bottom edge with $2m+2$, $2m+4$, . . ., $2m+2n-2$, $2(m+n)-1$.

We now raise the bottom left corner of the rectangle until the boundary nodes are at different heights, and so that the order of the heights matches the order of the labels (node 0 becomes the highest point, followed by nodes 1, 2, . . . in order). Fig. 17 illustrates the case $BC_{GP}(1,2)$. The black strut raises the bottom left corner so that the heights of the nodes are in the correct order.

We now take a horizontal line (Freeman calls it a "counting line"), and slide it downwards from the top of the figure to the bottom, recording each time it cuts a new cell. The counting line is shown in red in the figure.

When the counting line reaches a boundary node, with label k (say), the count is increased by the number of cells originating at k that have not yet been counted. This number is equal to the number of boundary nodes with label greater than k which are not on the same side as k. On the other hand, when the counting line reaches an interior node the count increases by exactly 1 (this is because there is no point where three chords meet). So the contribution to the count from the interior nodes is simply the number of interior nodes, which is known from Theorem 7.1.

In Fig. 17, the count goes up by 3 at node 0, by 3 at node 1, 1 at node 2, and 1 at node 3, for a subtotal of 8. There are 9 interior nodes, so the total number of cells is 17.

From a careful study of a tilted version of general case $BC_{GP}(m,n)$,, combined with (7.2), we obtain:

Theorem 7.2. *For $m, n \geq 1$, the number of cells in $BC_{GP}(m,n)$ is*

$$\mathcal{C}_{GP}(m,n) \;=\; \frac{1}{4}\{(m-1)^2(m-2)^2+(n-1)^2(n-2)^2\}+2mn\left(m+n-\frac{3}{2}\right)^2+\frac{9mn}{2}-1\,. \tag{7.5}$$

Remark. Asymptotically, $\mathcal{C}_{GP}(m,n)$ and $\mathcal{N}_{GP}(m,n)$ behave in the same way. In fact the difference $\mathcal{C}_{GP}(m,n) - \mathcal{N}_{GP}(m,n)$ is only a quadratic function of m and n, $m^2+4\,mn+n^2-4\,m-4\,n+1$.

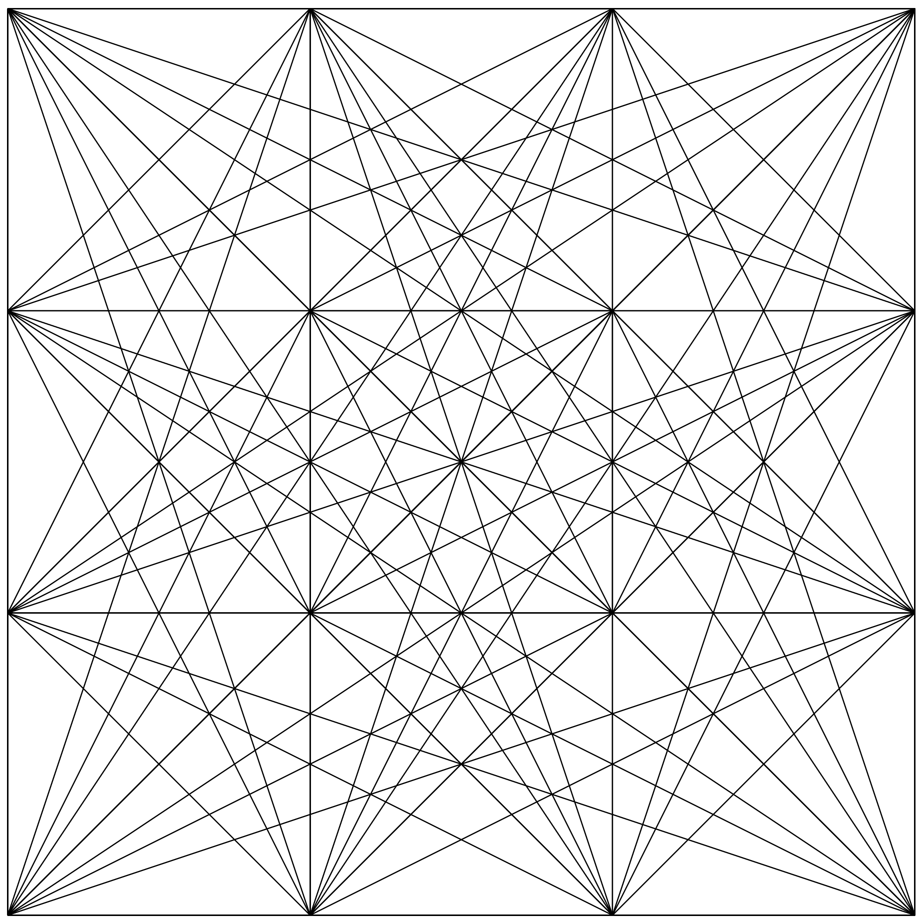


Figure 18: The graph $AC(3,3)$. There are 353 nodes and 520 cells.

8 The graphs $AC(m,n)$.

The graph $AC(m,n)$ was defined in §6. We take an $(m+1)\times(n+1)$ square grid of nodes, and draw a chord between *every* pair of grid nodes. (If we only joined pairs of boundary nodes we would get $BC(m,n)$.)

Figure 13 shows $AC(3,2)$ (take the black and red lines only, not the blue lines). Hugo Pfoertner has made black and white drawings of $AC(m,n)$ for $1 \leq m,n \leq 5$ in A288187. Figure 18 shows a black and white drawing of $AC(3,3)$ made using *TikZ* [5, 23].

The numbers of nodes $\mathcal{N}_{AC}(m,n)$ and cells $\mathcal{C}_{AC}(m,n)$ in $AC(m,n)$ are given for $m,n \le 9$ in A288180 and A288187, respectively, and the initial terms are shown in Table 11. The first row and column of Table 11 are the same as the first row and column of Table 7 but are included for completeness.

Table 11: Numbers of nodes $\mathcal{N}_{AC}(m,n)$ and cells $\mathcal{C}_{AC}(m,n)$ in $AC(m,n)$ for $1 \le m,n \le 7$.

$m\backslash n$	1	2	3	4	5	6	7
1	5, 4	13, 16	35, 46	75, 104	159, 214	275, 380	477, 648
2	13, 16	37, 56	121, 176	265, 388	587, 822	1019, 1452	1797, 2516
3	35, 46	121, 176	353, 520	771, 1152	1755, 2502	3075, 4392	5469, 7644
4	75, 104	265, 388	771, 1152	1761, 2584	4039, 5700	7035, 9944	12495, 17380
5	159, 214	587, 822	1755, 2502	4039, 5700	8917, 12368	15419, 21504	27229, 37572
6	275, 380	1019, 1452	3075, 4392	7035, 9944	15419, 21504	26773, 37400	47685, 65810
6	477, 648	1797, 2516	5469, 7644	12495, 17380	27229, 37572	47685, 65810	84497, 115532

It is clear (compare Figs. 14 and 18) that $AC(m,n)$ contains far more nodes and cells than $BC(m,n)$. We may obtain an upper bound on $\mathcal{N}_{AC}(n,n)$ as follows. The graph $AC(n,n)$ has $(n+1)^2$ grid points. The number of ways of choosing four grid points is $\binom{(n+1)^2}{4}$, and except for a vanishingly small fraction of cases, no three points will be collinear. There are then two possibilities: the four points may form a convex quadrilateral, or a triangle with the fourth point in its interior. In the first case the intersection of the two diagonals of the quadrilaterals is a node of $AC(m,n)$ (which may or may not be a new node), but in the second case no new node is formed.

If four points in the plane are chosen at random from a square, by what is known as "Sylvester's Theorem", the probability that they form a convex quadrilateral is 25/36 and the probability that they form a triangle with an interior point is 11/36 (see [16, Table 4], [21, Table 3, p. 114] for the complicated history of this result). Assuming that Sylvester's theorem applies to our problem, we can conclude that the number of nodes in $AC(m,n)$ counted with multiplicity is asymptotically

$$\frac{25}{36}\binom{(n+1)^2}{4} \sim \frac{1}{35.56}\, n^8 \,. \tag{8.1}$$

Both Tom Duff (personal communication) and Keith F. Lynch (personal communication) have have carried out extensive experiments, studying what happens when four points are chosen from an $m \times n$ grid, and have found that there is excellent agreement with the predictions of Sylvester's Theorem.

In a remarkable calculation, Tom Duff enumerated and classified all sets of four points chosen from an $m \times n$ grid for $m,n \le 349$. In a 349×349 grid, there are 6366733094048270910 strictly convex quadrilaterals out of 9170030499095875150 total. The fraction is 0.6942979, just a little short of Sylvester's $25/36 = 0.694444\ldots$. The deficit is explained by the not quite negligible counts of quadrilaterals with at least three collinear points. If those are included with the strictly convex quadrilaterals, the ratio is 0.6945982, slightly more than $25/36$. This is convincing evidence that Sylvester's theorem does apply to our problem.

In any case, $\mathcal{N}_{AC}(n,n) = O(n^8)$, compared with $\mathcal{N}(n,n) = O(n^4)$ for $BC(n,n)$.

9 The graphs $LC(m,n)$.

The graph $LC(m,n)$ was defined in §6. We take an $(m+1) \times (n+1)$ square grid of nodes, draw a chord between *every* pair of grid nodes, and extend all the chords until they meet the boundary of the grid. These

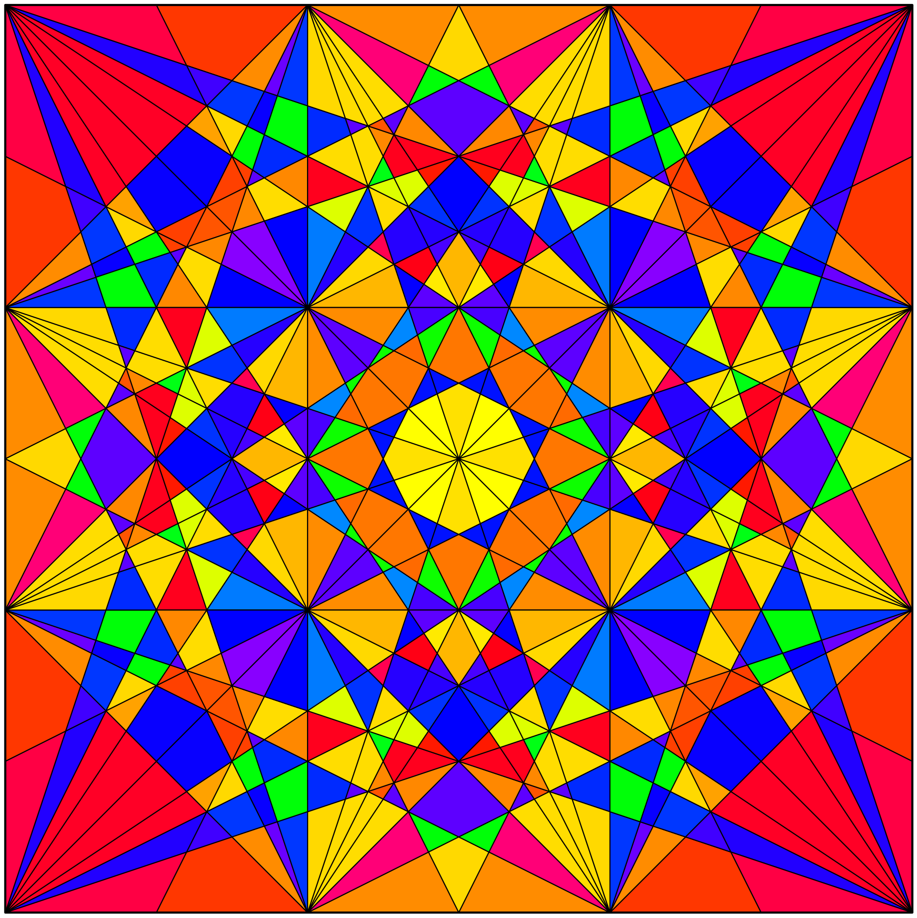


Figure 19: The graph $LC(3,3)$. There are 405 nodes and 624 cells.

graphs were discussed by Mustonen [11, 12, 13]. Figure 13 shows $LC(3,2)$ (take the black, red, and blue lines), and Fig. 19 shows our stained glass coloring of $LC(3,3)$.

The numbers of nodes $\mathcal{N}_{LC}(m,n)$ and cells $\mathcal{C}_{LC}(m,n)$ in $LC(m,n)$ are given for $m,n \le 8$ in A333284 and A333282, respectively, and the initial terms are shown in Table 12. Again the first row and column are the same as in Table 7. Mustonen [12, Table 3] gives the first 29 terms of the diagonal sequence $\mathcal{N}_{LC}(n,n)$ (A333285).

For this problem we can give only an upper bound on the number of nodes counted with multiplicity. The argument does, however, avoid the use of Sylvester's Theorem. Consider four points chosen from the $(n+1)\times(n+1)$ grid points, with no three points collinear. If the points form a triangle with a point in the interior, joining the three vertices of the triangle to the interior point and then extending these chords until they meet the sides of the triangle (something we were not allowed to do in the previous case) will produce three potentially new nodes. If the four points form a convex quadrilateral, there are also potentially three nodes that could be created: the intersection of the two diagonals, and the two points where pairs of opposite sides meet when extended. Figure 20 shows the two cases. The black nodes are the four grid points and the red nodes are the potential new nodes. Of course in the second case the two external red points may be outside the grid (or at infinity), and so would not be counted.

Table 12: Numbers of nodes $\mathcal{N}_{LC}(m,n)$ and cells $\mathcal{C}_{LC}(m,n)$ in $LC(m,n)$ for $1 \le m,n \le 7$.

$m\backslash n$	1	2	3	4	5	6	7
1	5, 4	13, 16	35, 46	75, 104	159, 214	275, 380	477, 648
2	13, 16	37, 56	129, 192	289, 428	663, 942	1163, 1672	2069, 2940
3	35, 46	129, 192	405, 624	933, 1416	2155, 3178	3793, 5612	6771, 9926
4	75, 104	289, 428	933, 1416	2225, 3288	5157, 7520	9051, 13188	16129, 23368
5	159, 214	663, 942	2155, 3178	5157, 7520	11641, 16912	20341, 29588	36173, 52368
6	275, 380	1163, 1672	3793, 5612	9051, 13188	20341, 29588	35677, 51864	63987, 92518
7	477, 648	2069, 2940	6771, 9926	16129, 23368	36173, 52368	63987, 92518	114409, 164692

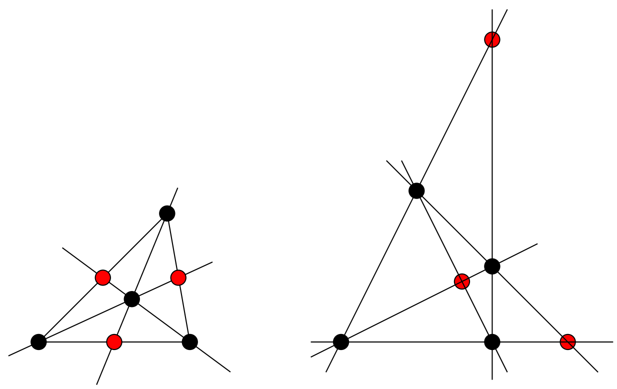


Figure 20: The two possibilities for choosing four noncollinear points from an $m \times n$ grid.

In any case, the maximum number of new nodes that are created is at most $3\binom{(n+1)^2}{4} \sim n^8/8$, and this is an upper bound on $\mathcal{N}_{LC}(n,n)$. This is an over-count, both because we do not always get three new nodes for each 4-tuple of grid points, and because multiple intersection points are counted multiple times. Based on his data for $n \le 29$, Mustonen [12] makes an empirical estimate that $\mathcal{N}_{LC}(n,n) \sim Cn^8$, where C is about 0.0075. So our constant, 1/8 is, unsurprisingly, an over-estimate.

We conclude that as we progress from $BC(n,n)$ to $AC(n,n)$ to $LC(n,n)$, the graphs become progressively more dense, and so counting the nodes with multiplicity gives a steadily weaker upper bound on their number.

10 Choosing the colors.

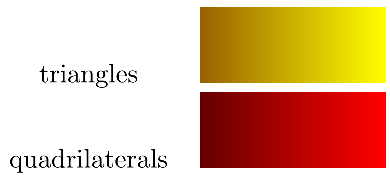
We used three different coloring schemes.

10.1 Number-of-sides coloring.

The simplest scheme colors the cells according to the number of sides, with randomly chosen colors. This is used in Fig. 15 and in figures in [14] (entries A333282, A335701, for example) when studying the distribution of cells according to number of sides.

10.2 The yellow and red palettes.

This is a refinement of the previous scheme, which modifies the color according to the shape of the cell. For Figs. 4, 5, 6, 7, 10, 11 the cells are either triangles or quadrilaterals, and we use colors which darken as the cell becomes more irregular. More precisely, the cells are colored according to the following rule. If the cell has n sides (where n is 3 or 4), let λ be the area of the cell divided by the area of an n-sided regular polygon with the same circumradius. Then the cell is assigned color number $\sqrt{\lambda}$ from the following palettes:



10.3 Random colorings.

For Figs. 1, 7, 14, etc. the color of a cell is assigned by first computing the average distance of the nodes of the cell from the center of the picture. These average distances are then grouped into a certain number of bins (we used 1000 bins), and the nonempty bins are assigned a random color from the standard spectrum from red to violet. This ensures a symmetrical coloring with contrasting colors for neighboring cells. In practice we do this several times and then choose the most appealing picture. We also have the option of restricting the color palette to achieve certain effects (reds, blues, and greens for a cathedral-like window, or various shades of browns for the frames that we will see in Part 2).

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References

[1] M. A. Alekseyev, On the number of two-dimensional threshold functions, *SIAM J. Discr. Math.*, **24:4** (2010), 1617–1631.

[2] M. A. Alekseyev, M. Basova, and N. Yu. Zolotykh, On the minimal teaching sets of two-dimensional threshold functions, *SIAM J. Discr. Math.*, **29:1** (2015), 157–165.

[3] L. Blomberg, S. R. Shannon, and N. J. A. Sloane, Graphical enumeration and stained glass windows, 2: Polygons, frames, crosses, etc., in preparation, 2020.

[4] B. Bollobás, *Graph Theory: An Introductory Course*, Springer, 1979.

[5] J. Crémer, A very minimal introduction to *TikZ*, March 11, 2011; https://cremeronline.com/LaTeX/minimaltikz.pdf.

[6] J. W. Freeman, The number of regions determined by a convex polygon, *Math. Mag.*, **49:1** (1975), 23–26.

[7] M. Griffiths, Counting the regions in a regular drawing of $K_{n,n}$, *J. Integer Sequences*, **13** (2010), #10.8.5.

[8] H. T. Hall, *Counterexamples in Discrete Geometry*, PhD Dissertation, Mathematics Department, University of California Berkeley, 2004.

[9] F. Harary, *Graph Theory*, Addison-Wesley, Reading MA, 1969.

[10] S. Legendre, The number of crossings in a regular drawing of the complete bipartite graph, *J. Integer Sequences*, **12** (2009), #09.5.5.

[11] S. Mustonen, Statistical accuracy of geometric constructions, September 2, 2008; http://www.survo.fi/papers/GeomAccuracy.pdf.

[12] S. Mustonen, On lines and their intersection points in a rectangular grid of points, April 16, 2009; http://www.survo.fi/papers/PointsInGrid.pdf.

[13] S. Mustonen, On lines going through a given number of points in a rectangular grid of points, May 12, 2010; http://www.survo.fi/papers/LinesInGrid2.pdf.

[14] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, 2020; https://oeis.org.

[15] M. E. Pfetsch and G. M. Ziegler, Large chambers in a lattice polygon, December 13, 2004; http://www.mathematik.tu-darmstadt.de/~pfetsch/chambers.

[16] R. E. Pfiefer, The historical development of J. J. Sylvester's Four Point Theorem, *Math. Mag.*, **62:5** (1989), 309–317.

[17] B. Poonen and M. Rubinstein, The number of intersection points made by the diagonals of a regular polygon, *SIAM J. Discr. Math.*, **11:1** (1998), 135–156.

[18] B. Salvy and P. Zimmermann, GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable, *ACM Trans. Math. Software*, **2**0 (1994), 163–177.

[19] V. N. Shevchenko and N. Yu. Zolotykh, On the complexity of deciphering the threshold functions of k-valued logic, (Russian) *Dokl. Akad. Nauk*, **362:5** (1998), 606–608; (English translation) *Dokl. Math.*, **58** (1998), 268-270.

[20] N. J. A. Sloane, The email servers and Superseeker, 2010; https://oeis.org/ol.html.

[21] H. Solomon, *Geometric Probability*, SIAM, Philadelphia, 1978.

[22] S. E. Sommars and T. Sommars, Number of triangles formed by intersecting diagonals of a regular polygon, *J. Integer Sequences*, **1** (1998), #98.1.5.

[23] T. Tantau, *The PGF/TikZ Programming Language*, Version 2.10, CTAN Org., October 25 2010.

[24] W. T. Tutte, A census of planar maps, *Canad. J. Math.*, **15** (1963), 249–271.

[25] W. T. Tutte, On the enumeration of planar maps, *Bull. Amer. Math. Soc.*, **74** (1968), 64–74.

[26] N. Yu. Zolotykh, On the complexity of deciphering threshold functions in two variables, (Russian), in *Proc. 11th Internat. School Seminar "Synthesis and complexity of control systems," Part I*, Center of Applied Research, Moscow State Univ. Faculty of Mechanics and Mathematics, Moscow, Russia, 2001, pp. 74-79.

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