Gray Codes for Reflection Groups

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Abstract. Let $G$ be a finite group generated by reflections. It is shown that the elements of $G$ can be arranged in a cycle (a "Gray code") such that each element is obtained from the previous one by applying one of the generators. The case $G = S_n$ yields a conventional binary Gray code. These generalized Gray codes provide an efficient way to run through the elements of any finite reflection group.

1. Introduction

The classical version of a Gray code is a Hamiltonian circuit through the $2^n$ vertices of the $n$-cube, or equivalently an ordering of the $2^n$ binary vectors of length $n$ such that each pair of adjacent vectors (including the first and last) differ in a single position. For the extensive literature see the bibliography. The first appearance of the "Gray code" that we have located is in 1872 [29].

As we will show, the classical version is the special case $G = S_n$ of the following.

Theorem. Let $G$ be a finite group generated by reflections $R_1, \ldots, R_m$. Then there is a Hamiltonian circuit in the Cayley diagram for $G$ corresponding to these generators. In other words the $g = |G|$ elements of $G$ can be arranged in order

$$\{a_0, a_1, \ldots, a_{g-1}\}$$

so that for each $i$ ($0 \leq i \leq g - 1$) there is $j$ so that $a_{i+1} = a_i R_j$ (where $a_g = a_0$).

We call (1) a Gray code for $G$.

It is well-known that any group generated by reflections can be described by a Coxeter diagram [7, 14, 15, 31]. The finite reflection groups for which the Coxeter diagram is a connected graph are ([7], p. 193, Theorem 1) the groups $S_n$ ($n \geq 1$), $B_n$ ($n \geq 2$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$, $F_4$, $G_2$, $H_3$, $H_4$ and $J_m$ ($m = 5$ or $m > 7$).* These are the irreducible reflection groups. Figure 1 shows their Coxeter diagrams,

* We follow Grove and Benson [31] in using script letters for these groups, to distinguish them from the Lie groups and Euclidean lattices with similar names.