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Four-Dimensional Modulation With an Eight-State Trellis Code

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A trellis code is a "sliding window" method for encoding a binary data stream $\{a^i\}$, $a^i = 0, 1$, as a sequence of signal points drawn from $\mathbf{R}^n$. The rule for assigning signal points depends on the state of the encoder. In this paper $n = 4$, and the signal points are 4-tuples of odd integers. We describe an infinite family of eight-state trellis codes. For $k = 3, 4, 5, \cdots$ we construct a trellis encoder with a rate of k bits/four-dimensional signal. We propose that the codes with rates $k = 8$ and 12 be considered for use in modems designed to achieve data rates of 9.6 kb/s and 14.4 kb/s, respectively.

I. INTRODUCTION

A trellis code is a "sliding window" method for encoding a binary data stream $\{a^i\}$, $a^i = 0, 1$, as a sequence of signal points $\{\mathbf{x}^i\}$ drawn from $\mathbf{R}^n$. The set of possible signal points is finite, and this set is called the *signal constellation.* The purpose of coding is to gain noise immunity beyond that provided by standard uncoded transmission at the same data rate. In this paper $n = 4$, and the signal points are drawn from $(2\mathbf{Z} + 1)^4$, the lattice of 4-tuples of odd integers. We shall regard transmission of a four-dimensional signal as one use of the channel, and we measure the rate of the code in bits per channel use. The four-dimensional signal space can be realized by using two space-orthogonal electric field polarizations to communicate on the same carrier frequency. It is also possible to regard each four-dimensional symbol as two consecutive two-dimensional symbols.

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Ungerboeck[1] described a technique called *set partitioning*, which assigns signal points to successive blocks of input data. The rule for assigning signal points depends on the state of the encoder. Ungerboeck constructed simple *trellis codes* providing the same noise immunity as is given by increasing the power of uncoded transmission by factors ranging from 2 to 4 (coding gains ranging from 3 to 6 dB). Calderbank and Mazo[2] have given a different algebraic description of trellis codes. Trellis codes with a rate of 4 bits/two-dimensional symbol have recently been proposed for use in modems designed to achieve data rates of 9.6 kb/s on dial-up voice telephone lines. These codes use the signal constellation shown in Fig. 1, which was originally described by Campopiano and Glazer,[3] and gain 4 dB over uncoded transmission at the same rate. In Section III of this paper we describe the first code in our infinite family. This code has a rate of 8 bits/four-dimensional symbol and promises a gain of 4.7 dB over uncoded transmission. The signal constellation consists of 512 four-dimensional signal points. Transmission of two consecutive two-dimensional signals using one of the proposed trellis codes with a rate of 4 bits/two-dimensional symbol requires $1024 = 32^2$ four-dimensional signal points. Furthermore, the restriction of the 512-point constellation to the first two coordinates, or to the last two coordinates, is the 32-point constellation shown in Fig. 1. The 0.7-dB improvement in performance is derived from reducing the average transmitted power.

In Section IV we briefly describe the second code in the family, which has a rate of 12 bits/four-dimensional symbol and promises a

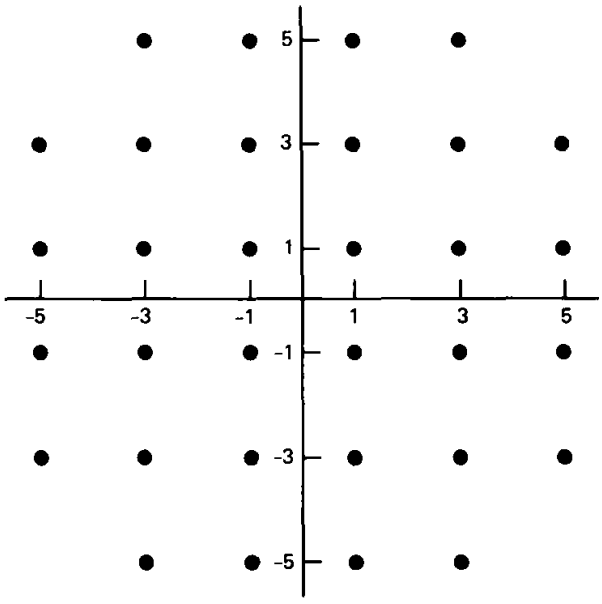

Fig. 1—Signal constellation for proposed trellis codes with rate 4 bits/two-dimensional symbol.

coding gain of 4.9 dB over uncoded transmission. We propose using this code in modems designed to achieve data rates of 14.4 kb/s.

We begin by presenting a rate 3/4 binary convolutional code that is basic to the construction of the new trellis codes. In Section V we describe the general code in the family, with rate k bits/four-dimensional symbol, for $k = 3, 4, 5 \cdots$, and we show that in the limit, as $k \to \infty$, the coding gain is asymptotic to $10 \log_{10}\pi \approx 4.9715$ dB. The difference between this limiting coding gain and that provided by the code with $k = 12$ is very small.

After this paper was submitted, we discovered that Forney et al.[4] had independently proposed a different rate $k = 8$ code with approximately the same performance. Wilson, Sleeper, and Smith[5] have described simple trellis codes (with up to four encoder states) that use four-dimensional signal constellations.

II. A RATE 3/4 BINARY CONVOLUTIONAL CODE

We assume that binary data are being encoded at a rate of k bits/signal point and that the data enter the encoder in k parallel sequences, $\{a_1^i\}, \{a_2^i\}, \cdots, \{a_k^i\}$. We assume that the output $\mathbf{x}^i$ of the trellis encoder at time i depends not only on the present values $a_1^i, a_2^i, \cdots, a_k^i$, of the input sequences, but also on the previous $\nu_j \geq 0$ bits of the jth sequence. If $\nu_j = 0$ for some j, then $a_j^1, a_j^2, a_j^3, \cdots$ is said to be a sequence of *uncoded bits*. The *constraint length* ν is given by $\nu = \sum_{j=1}^{k} \nu_j$. The output $\mathbf{x}^i$ of the encoder is a fixed vector-valued function $\mathbf{x}$ of the $\nu + k$ binary variables $a_1^i, \cdots, a_1^{i-\nu_1}; a_2^i, \cdots, a_2^{i-\nu_2}; \cdots; a_k^i, \cdots, a_k^{i-\nu_k}$. That is,

$$\mathbf{x}^i = \mathbf{x}(a_1^i a_1^{i-1} \cdots a_1^{i-\nu_1}; \cdots; a_k^i \cdots a_k^{i-\nu_k}).$$

The ν-tuple $(a_1^{i-1} \cdots a_1^{i-\nu_1}; a_2^{i-1} \cdots a_2^{i-\nu_2}; \cdots; a_k^{i-\nu_k})$ is the *state* of the encoder and there are 2^ν states. Figure 2 shows a state transition diagram for a trellis code with $k = 3$, $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 2$. The average transmitted signal power P is given by

$$P = \frac{1}{2^{k+\nu}} \sum \| \mathbf{x}(a_1^i \cdots a_1^{i-\nu_1}; \cdots; a_k^i \cdots a_k^{i-\nu_k}) \|^2.$$

Basic to the trellis codes constructed below is a certain rate 3/4 binary convolutional code with total memory 3 and free distance 4. The encoder is presented in Fig. 3, which is taken from Ref. 6 (Fig. 10.3, p. 292). The three parallel input sequences determine the output sequence $\{\mathbf{v}^i = (v_1^i, v_2^i, v_3^i, v_4^i)\}$ according to the following rules:

$$v_1^i = a_1^i,$$

$$v_2^i = a_1^i + a_2^i + a_2^{i-1} + a_3^{i-1},$$

$$v_3^i = a_1^i + a_2^{i-1} + a_3^i + a_3^{i-2},$$

$$v_4^i = a_1^i + a_2^i + a_3^i + a_3^{i-2}.$$

The triple $a_2^{i-1}a_3^{i-1}a_3^{i-2}$ is the state of the encoder. The possible transitions between states are shown in Fig. 2. The edge joining state $a_2^{i-1}a_3^{i-1}a_3^{i-2}$ to state $a_2^i a_3^i a_3^{i-1}$ is labeled with the outputs $\mathbf{v}^i = (v_1^i, v_2^i, v_3^i, v_4^i)$ and $\bar{\mathbf{v}}^i = (\bar{v}_1^i, \bar{v}_2^i, \bar{v}_3^i, \bar{v}_4^i)$ corresponding to this transition. Note that

$$\bar{v}_1^i = \bar{a}_1^i,$$

$$\bar{v}_2^i = \bar{a}_1^i + a_2^i + a_2^{i-1} + a_3^{i-1},$$

$$\bar{v}_3^i = \bar{a}_1^i + a_2^{i-1} + a_3^i + a_3^{i-2},$$

$$\bar{v}_4^i = \bar{a}_1^i + a_2^i + a_3^i + a_3^{i-2}.$$

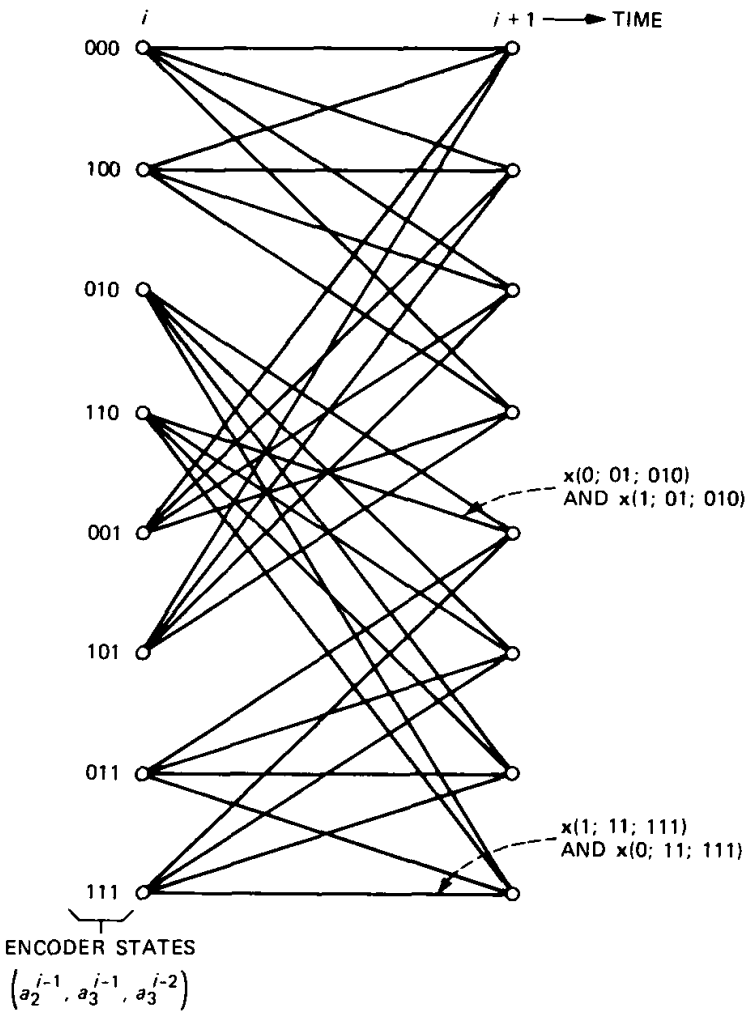

Fig. 2—A state transition diagram for a trellis code with $k = 3$, $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 2$. (Every edge represents two possible transitions.)

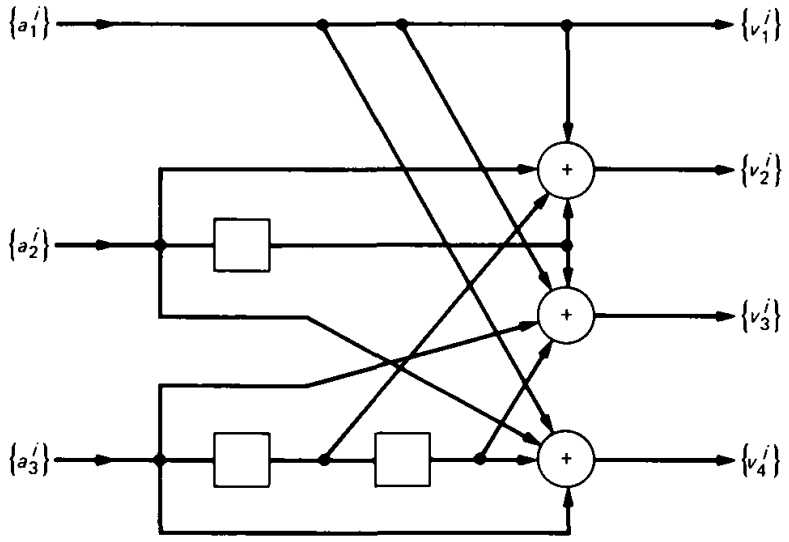

Fig. 3—A rate 3/4 binary convolutional code.

We change from 0, 1 notation to ±1 notation ($0 \leftrightarrow +1$ and $1 \leftrightarrow -1$). An edge joining two states is now labeled with pairs of vectors $\pm(w_1, w_2, w_3, w_4)$, where $w_i = \pm 1$, $i = 1, 2, 3, 4$. This defines a trellis encoder with a rate of 3 bits/four-dimensional symbol. The minimum squared distance of this trellis code is simply four times the free distance of the original binary convolutional code, namely, 16. This is because 0 opposite 1 contributes 1 to the free distance, whereas 1 opposite −1 contributes 4 to the squared minimum distance.

Transmission at the higher rates of 8 and 12 bits/four-dimensional symbol requires more channel symbols. Indeed, to achieve any coding gain, we have to use more symbols than are required by uncoded transmission at the same rate.

III. A TRELLIS CODE WITH RATE 8 BITS/FOUR-DIMENSIONAL SYMBOL

Uncoded transmission at the rate 4 bits/two-dimensional symbol uses the rectangular signal constellation shown in Fig. 4. To achieve uncoded transmission of a four-dimensional symbol at a rate of 8 bits/symbol, simply take two copies of this scheme. There are 256 possible signals and the average power is $4(1^2 + 3^2)/2 = 20$. Since the minimum squared distance between distinct signals is 4, we have

$$\left(\frac{d^2_{\min}}{P}\right)_{\text{uncoded}} = \frac{4}{20}.$$

For coded transmission we shall use $2 \times 256 = 512$ signal points. Representative signal points are listed in Table I. The remaining points are obtained from these representatives by permuting the

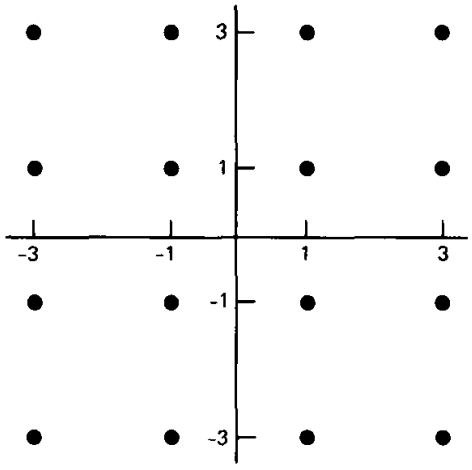

Fig. 4—The rectangular constellation for uncoded transmission at 4 bits/two-dimensional symbol.

Table I—The signal constellation for coded transmission at 8 bits/four-dimensional symbol. All permutations of coordinates and all sign changes are allowed.

Representative	Energy	1/16 × Number
(1111)	4	1
(3111)	12	4
(3311)	20	6
(5111)	28	4
(3331)	28	4
(5311)	36	12
(3333)	36	1

coordinates and changing signs in all possible ways. For example, $(3\bar{1}\bar{3}1)$, $(131\bar{5})$, and $(\bar{5}111)$ are all signal points (where $\bar{x}$ denotes $-x$). For every vector $\mathbf{w} = (w_1, w_2, w_3, w_4)$ with $w_i = \pm 1$, $i = 1, 2, 3, 4$, let $S(\mathbf{w})$ be the set of 32 signal points (x_1, x_2, x_3, x_4) satisfying $x_i \equiv w_i \pmod 4$, for $i = 1, 2, 3, 4$. The sets $S(\mathbf{w})$ partition the signal constellation into 16 equal parts. The set $S(1111)$ is shown in Table II and the other sets are obtained from $S(1111)$ by changing signs. For example, $S(1\bar{1}1\bar{1})$ is obtained from $S(1111)$ by changing the signs of the second and fourth entries. The distance $d(A, B)$ between two sets of vectors A and B is given by

$$d(A, B) = \min_{\mathbf{x}\in A, \mathbf{y}\in B} \{\|\mathbf{x} - \mathbf{y}\|\}.$$

The partition into sets $S(\mathbf{w})$ satisfies the following metric properties:

Table II—The set $S(1111)$. All permutations of coordinates are allowed.

Signal Point	Energy	Number
(1111)	4	1
$(\bar{3}111)$	12	4
$(\bar{3}\bar{3}11)$	20	6
$(5\bar{1}11)$	28	4
$(\bar{3}\bar{3}\bar{3}1)$	28	4
$(5\bar{3}\bar{1}1)$	36	12
$(\bar{3}\bar{3}\bar{3}\bar{3})$	36	1

(M1) if $\mathbf{x}, \mathbf{y} \in S(\mathbf{w})$ then $\|\mathbf{x} - \mathbf{y}\|^2 \geq 16$,

(M2) if $\mathbf{v} \neq \mathbf{w}$ then $d^2(S(\mathbf{v}), S(\mathbf{w})) = \|\mathbf{v} - \mathbf{w}\|^2$.

To verify (M1) let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$. Then $x_i \neq y_i$ for some i. Since $x_i \equiv y_i \pmod 4$, we have $\|\mathbf{x} - \mathbf{y}\|^2 \geq 16$. To verify (M2) let $\mathbf{x} = (x_1, x_2, x_3, x_4) \in S(\mathbf{v})$ and $\mathbf{y} = (y_1, y_2, y_3, y_4) \in S(\mathbf{w})$. If $x_i \neq y_i \pmod 4$ then $|x_i - y_i|^2 \geq 4$. Hence $\|\mathbf{x} - \mathbf{y}\|^2 \geq \|\mathbf{v} - \mathbf{w}\|^2$ and equality holds when $\mathbf{x} = \mathbf{v}$ and $\mathbf{y} = \mathbf{w}$.

In Section II we described a trellis code with a rate of 3 bits/four-dimensional symbol and minimum squared distance $d^2_{\min} = 16$. To achieve the higher transmission rate of 8 bits/four-dimensional symbol, we add 5 uncoded bits. There are now eight parallel input sequences $\{a_1^i\}, \cdots, \{a_8^i\}$. The sequences $\{a_2^i\}, \{a_3^i\}$ determine the state $a_2^{i-1}a_3^{i-1}a_3^{i-2}$ of the encoder as in Fig. 3. An edge joining two states that was originally labeled by the pair of vectors $\pm\mathbf{v}$ is now labeled by the 64 vectors in $S(\mathbf{v}) \cup S(-\mathbf{v})$. This is because there are 64 parallel transitions between states $a_2^{i-1}a_3^{i-1}a_3^{i-2}$ and $a_2^i a_3^i a_3^{i-1}$ corresponding to the 64 possible inputs $a_1^i a_4^i \cdots a_8^i$. We allow any fixed assignment of channel symbols in $S(\mathbf{v}) \cup S(-\mathbf{v})$ to inputs $a_1^i a_4^i \cdots a_8^i$.

Consider the distance properties of the high-rate code. Properties (M1) and (M2) guarantee that the squared distance of any error event of length 1 is at least 16. Consider any error event in the eight-state trellis of length greater than 1. If the squared distance for the low-rate code is

$$\sum_{i=1}^{l} \|\mathbf{v}^i - \hat{\mathbf{v}}^i\|^2,$$

then the squared distance for the high-rate code is at least

$$\sum_{i=1}^{l} d^2(S(\mathbf{v}^i), S(\hat{\mathbf{v}}^i)).$$

Property (M2) now implies that the minimum squared distance of the high-rate code is at least 16.

The average signal power P of the 512-point signal constellation is given by

$$P = \frac{16(4 \times 1 + 12 \times 4 + 20 \times 6 + 28 \times 8 + 36 \times 13)}{512} = 27.$$

Thus,

$$\left(\frac{d^2_{\min}}{P}\right)_{\text{coded}} = \frac{16}{27},$$

and the coding gain (in decibels) is

$$10 \log_{10}\left(\frac{(d^2_{\min}/P)_{\text{coded}}}{(d^2_{\min}/P)_{\text{uncoded}}}\right) = 10 \log_{10}\left(\frac{16/27}{4/20}\right) \approx 4.717 \text{ db}.$$

IV. A TRELLIS CODE WITH RATE 12 BITS/FOUR-DIMENSIONAL SYMBOL

Uncoded transmission at the rate of 6 bits/two-dimensional symbol uses the 64-point rectangular constellation shown in Fig. 5. To achieve uncoded transmission of a four-dimensional symbol at a rate of 12 bits/symbol, simply take two copies of this scheme. There are $64^2 = 2^{12}$ possible signals, and the average signal power P is $4(1^2 + 3^2 + 5^2 + 7^2)/4 = 84$. Thus,

$$\left(\frac{d^2_{\min}}{P}\right)_{\text{uncoded}} = \frac{4}{84}.$$

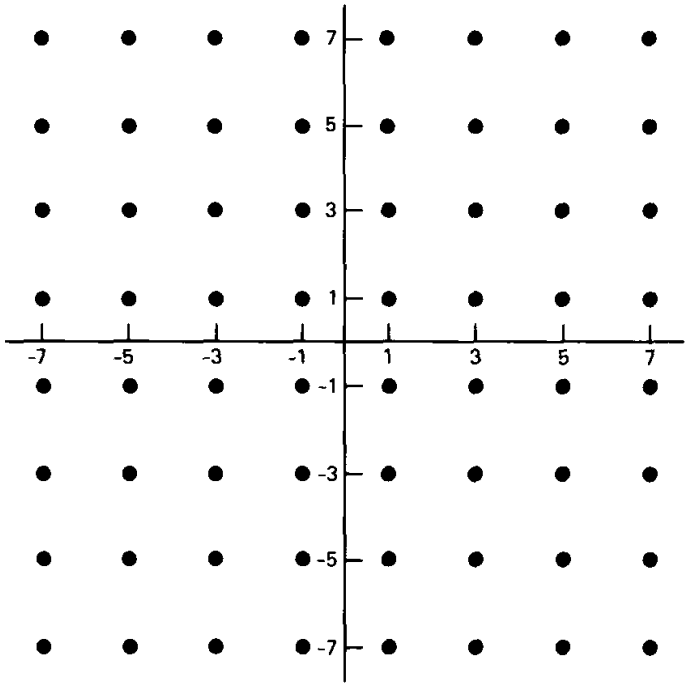

Fig. 5—A rectangular constellation for uncoded transmission at 6 bits/two-dimensional symbol.

Table III—The signal constellation for coded transmission at 12 bits/four-dimensional symbol. All representatives are taken from $S(1111)$.

Representative	Energy	(1/16) × Number
(1111)	4	1
$(\bar{3}111)$	12	4
$(\bar{3}\bar{3}11)$	20	6
$(5111), (\bar{3}\bar{3}\bar{3}1)$	28	4 + 4 = 8
$(5\bar{3}11), (\bar{3}\bar{3}\bar{3}\bar{3})$	36	12 + 1 = 13
$(5\bar{3}\bar{3}1)$	44	12
$(\bar{7}111), (5\bar{3}\bar{3}\bar{3}), (5511)$	52	4 + 4 + 6 = 14
$(\bar{7}\bar{3}11), (55\bar{3}1)$	60	12 + 12 = 24
$(\bar{7}\bar{3}\bar{3}1), (55\bar{3}\bar{3})$	68	12 + 6 = 18
$(\bar{7}\bar{3}\bar{3}\bar{3}), (5551), (\bar{7}511)$	76	4 + 4 + 12 = 20
$(9111), (\bar{7}5\bar{3}1), (555\bar{3})$	84	4 + 24 + 4 = 32
$(9\bar{3}11), (\bar{7}5\bar{3}\bar{3})$	92	12 + 12 = 24
$(9\bar{3}\bar{3}1), (\bar{7}\bar{7}11), (\bar{7}551), (5555)$	100	12 + 6 + 12 + 1 = 31
$(\bar{7}553), (9\bar{3}\bar{3}\bar{3}), (9511), (\bar{7}\bar{7}\bar{3}1)$	108	12 + 4 + 12 + 12 = 40
$(95\bar{3}1), (\bar{7}\bar{7}\bar{3}\bar{3})$	116	24 + 6 = 30
$(95\bar{3}\bar{3}), (\bar{1}\bar{1}111), (\bar{7}\bar{7}51), (\bar{7}555)$	124	12 + 4 + 12 + 4 = 32
$(\bar{1}\bar{1}\bar{3}11), (9\bar{7}11), (9551), (\bar{7}\bar{7}5\bar{3})$	132	12 + 12 + 12 + 12 = 48
$(\bar{1}\bar{1}\bar{3}\bar{3}1), (9\bar{7}\bar{3}1), (955\bar{3})$	140	12 + 24 + 12 = 48
$(\bar{1}\bar{1}\bar{3}\bar{3}\bar{3}), (\bar{1}\bar{1}511), (9\bar{7}\bar{3}\bar{3}), (\bar{7}\bar{7}55), (\bar{7}\bar{7}\bar{7}1)$	148	4 + 12 + 12 + 6 + 4 = 38
$(\bar{1}\bar{1}5\bar{3}1), (9555), (9\bar{7}51), (\bar{7}\bar{7}\bar{7}3)$	156	24 + 4 + 24 + 4 = 56
$(\bar{1}\bar{1}5\bar{3}\bar{3}), (9911), (9\bar{7}5\bar{3})$	164	only 13

For coded transmission we use $2 \times 2^{12} = 2^{13}$ signal points. As in Section III we partition the signal constellation into 16 sets $S(\mathbf{w})$ according to congruence of the entries modulo 4. Each set $S(\mathbf{w})$ contains 512 signal points. Representative signal points are listed in Table III, where the representatives are all taken from $S(1111)$.

To achieve the transmission rate of 12 bits/four-dimensional symbol, we add 9 uncoded bits to the low-rate trellis code described in Section II. There are now 1024 parallel transitions between states $a_2^{i-1}a_3^{i-1}a_3^{i-2}$ and $a_2^i a_3^i a_3^{i-1}$ in the eight-state trellis. If the edge corresponding to this transition was originally labeled $\pm\mathbf{v}$, it is now labeled with the 1024 vectors in $S(\mathbf{v}) \cup S(-\mathbf{v})$. The metric properties (M1) and (M2) guarantee that the squared minimum distance of the high-rate code is equal to the squared minimum distance of the low-rate code, which is 16. An easy calculation shows that the average signal power P is 108.625, so

$$\left(\frac{d_{\min}^2}{P}\right)_{\text{coded}} = \frac{16}{108.625}.$$

The coding gain is

$$10 \log_{10}\left(\frac{10/108.625}{4/84}\right) \approx 4.904 \text{ dB}.$$

V. ASYMPTOTIC PERFORMANCE OF A FAMILY OF CODES

To achieve coded transmission at the rate of k bits/four-dimensional signal, we add $k-3$ uncoded bits to the low-rate trellis code described in Section II. There are 2^{k-2} parallel transitions between states $a_2^{i-1}a_3^{i-1}a_3^{i-2}$ and $a_2^i a_3^i a_3^{i-1}$ in the eight-state trellis. Coded transmission requires 2^{k+1} signal points. The points of the lattice $(2\mathbf{Z}+1)^4$ lie in shells around the origin consisting of 16 vectors of energy 4, 64 vectors of energy 12, and so on (see Table III). The 2^{k+1} signal points are obtained by taking all points of energy 4, 12, 20, $\cdots$ and just enough points of a final shell to bring the total number up to 2^{k+1}. The signal constellation is partitioned into 16 sets $S(\mathbf{v})$ according to congruence of the entries modulo 4. Each set contains 2^{k-3} signal points. Edges in the eight-state trellis originally labelled $\pm\mathbf{w}$ are now labeled with the 2^{k-2} vectors in $S(\mathbf{v}) \cup S(-\mathbf{v})$. The metric properties (M1) and (M2) guarantee that the minimum squared distance of this trellis code is 16.

Consider the asymptotic performance of this family of codes. For simplicity suppose that the signal constellation of each code in the family is a complete union of energy shells. If $\mathbf{x}$ is a vector in the lattice $(2\mathbf{Z}+1)^4$, then $\|\mathbf{x}\|^2 \equiv 4 \pmod 8$, since $\|\mathbf{x}\|^2$ is the sum of four odd squares. A classical result, due to Jacobi and to Legendre, is that every positive integer of the form $8n+4$ is a sum of four odd squares in $\sigma(2n+1)$ ways, where $\sigma(m)$ is the sum of divisors of m. The generating function

$$16 \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \sigma(m) q^{4m}$$

expresses the fact that there are 16 $\sigma(m)$ vectors of energy $4m$ in the lattice $(2\mathbf{Z}+1)^4$. The factor of 16 arises from the 16 possible sign changes.

The following estimates (which are proved in the Appendix) will be used to calculate the average energy of the vectors $\mathbf{x}$ in $(2\mathbf{Z}+1)^4$ with $\|\mathbf{x}\|^2 \leq 4n$:

$$\sum_{\substack{1 \leq m \leq n \\ m \text{ odd}}} \sigma(m) = \frac{\pi^2 n^2}{32} + O(n \log n), \tag{1}$$

$$\sum_{\substack{1 \leq m \leq n \\ m \text{ odd}}} m\sigma(m) = \frac{\pi^2 n^3}{48} + O(n^2 \log n). \tag{2}$$

The average energy P of the vectors $\mathbf{x}$ in $(2\mathbf{Z}+1)^4$ with $\|\mathbf{x}\|^2 \leq 4n$ is given by

$$P = \frac{4(\pi^2 n^3/48 + O(n^2 \log n))}{\pi^2 n^2/32 + O(n \log n)} = \frac{8}{3} n + O(\log n).$$

Therefore,

$$\left(\frac{d^2_{\min}}{P}\right)_{\text{coded}} = \frac{16}{(8/3)n + O(\log n)} = \frac{6}{n} + O(\log n)/n^2. \tag{3}$$

The number of points in the signal constellation is

$$16 \sum_{\substack{1 \le m \le n \\ m \text{ odd}}} \sigma(m) = \frac{\pi^2 n^2}{2} + O(n \log n).$$

For uncoded transmission we use just half this many points. The signal constellation is the set of all 4-tuples $\mathbf{x} = (x_1, x_2, x_3, x_4)$, where $x_i = \pm 1, \pm 3, \cdots, \pm(2a - 1)$. The number of points in this constellation is $16a^4$, so we choose a to make a^4 close to $\pi^2 n^2/64$. Now

$$1^2 + 3^2 + 5^2 + \cdots + (2a - 1)^2 = \frac{a(4a^2 - 1)}{3},$$

so the average signal power P is given by

$$P = \frac{4a(4a^2 - 1)}{3} a = \frac{4}{3}(4a^2 - 1).$$

Since the minimum squared distance between distinct signals is 4, we have

$$\left(\frac{d^2_{\min}}{P}\right)_{\text{uncoded}} = \frac{4}{4(4a^2 - 1)/3} = \frac{6}{8a^2 - 2}.$$

Now $8a^2 - 2 = \pi n + O(\sqrt{n})$, so

$$\left(\frac{d^2_{\min}}{P}\right)_{\text{uncoded}} = \frac{6}{\pi n} + O(n^{-3/2}).$$

Therefore,

$$\lim_{n \to \infty} \frac{(d^2_{\min}/P)_{\text{coded}}}{(d^2_{\min}/P)_{\text{uncoded}}} = \pi,$$

and the limiting coding gain is $10 \log_{10} \pi = 4.9714 \cdots$ dB.

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APPENDIX

Proof of Equations (1) and (2)

To prove that

$$\sum_{\substack{1\leqslant m\leqslant n\\ m \text{ odd}}} \sigma(m) = \frac{\pi^2 n^2}{32} + O(n \log n),$$

we write

$$\begin{aligned}\sum_{\substack{1\leqslant m\leqslant n\\ m \text{ odd}}} \sigma(m) &= \sum_{\substack{1\leqslant m\leqslant n\\ m \text{ odd}}} \sum_{\substack{q\mid m\\ q \text{ odd}}} q \\ &= \sum_{\substack{d\leqslant n\\ d \text{ odd}}} \sum_{\substack{q\leqslant n/d\\ q \text{ odd}}} q \\ &= \sum_{\substack{d\leqslant n\\ d \text{ odd}}} \frac{1}{4}\left((n/d)_0 + 1\right)^2,\end{aligned}$$

where $(n/d)_0$ is the largest odd integer $\leqslant n/d$. Then $(n/d)_0 + 1 = n/d + \mu$, where $-1 \leqslant \mu \leqslant 1$, and

$$\begin{aligned}\sum_{\substack{1\leqslant m\leqslant m\\ m \text{ odd}}} \sigma(m) &= \frac{n^2}{2}\left(\sum_{\substack{d\leqslant n\\ d \text{ odd}}} \frac{1}{d^2}\right) + O\left(n \sum_{\substack{d\leqslant n\\ d \text{ odd}}} \frac{1}{d}\right) \\ &= \frac{n^2}{4}\left(\sum_{\substack{d\leqslant n\\ d \text{ odd}}} \frac{1}{d^2}\right) + O(n \log n) \\ &= \frac{n^2}{4}\left(\left(\sum_{d\leqslant n} \frac{1}{d^2}\right) - \left(\sum_{\substack{d\leqslant n\\ d \text{ even}}} \frac{1}{d^2}\right)\right) + O(n \log n) \\ &= \frac{n^2}{4}\left(\left(\sum_{d\leqslant n} \frac{1}{d^2}\right) - \frac{1}{4}\left(\sum_{d\leqslant n/2} \frac{1}{d^2}\right)\right) + O(n \log n)\end{aligned}$$

$$= \frac{n^2}{4}\left(\left(\frac{\pi^2}{6} + O(1/n)\right) - \frac{1}{4}\left(\frac{\pi^2}{6} + O\,(1/n)\right)\right)$$
$$+ O(n \log n)$$
$$= \frac{n^2\pi^2}{32} + O(n \log n).$$

The estimates for partial sums are obtained using Euler's summation formula (see Ref. 7, p. 54). To prove

$$\sum_{\substack{1 \le m \le n \\ m \text{ odd}}} m\sigma(m) = \frac{\pi^2 n^3}{48} + O(n^2 \log n),$$

we write

$$\sum_{\substack{1 \le m \le n \\ m \text{ odd}}} m\sigma(m) = \sum_{\substack{d \le n \\ d \text{ odd}}} d \sum_{\substack{qd \le n \\ q \text{ odd}}} dq$$
$$= \sum_{t=1}^{(n-1)/2} \{1 + 3 + 5 + \cdots + (2t-1)\}$$
$$\cdot \sum_{\substack{n/(2t+1) < d \le n/(2t-1) \\ d \text{ odd}}} d^2$$
$$= \sum_{t=1}^{(n-1)/2} t^2 \sum_{\substack{n/(2t+1) < d \le n/(2t-1) \\ d \text{ odd}}} d^2$$
$$= \sum_{t=1}^{(n-1)/2} [t^2 - (t-1)^2] \sum_{\substack{d \le n/(2t-1) \\ d \text{ odd}}} d^2$$
$$= \sum_{t=1}^{(n-1)/2} (2t-1)$$
$$\cdot \left[\frac{1}{6}\frac{n}{2t-1}\left(\frac{n}{2t-1} + 1\right)\left(\frac{n}{2t-1} + 2\right)\right].$$

The coefficient of n^3 is

$$\frac{1}{6} \sum_{\substack{d \le (n-1)/2 \\ d \text{ odd}}} \frac{1}{d^2} = \frac{\pi^2}{48} + O(1/n),$$

so

$$\sum_{\substack{1 \le m \le n \\ m \text{ odd}}} m\sigma(m) = \frac{\pi^2 n^3}{48} + O(n^2 \log n)$$

as required.

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